B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 1

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Exercise 1. $\mathbb{CP}^1$ as a quotient of spheres.

Recall that the complex projective space $\mathbb{C}P^1$ is the space of complex lines through 0 in $\mathbb{C}^2$. By thinking about how a complex 1-dimensional vector space intersects the sphere $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$, show that $\mathbb{C}P^1$ as a topological surface can be viewed as a quotient

$$\mathbb{C}P^1 = S^3/S^1$$

where you need to explain how the group S^1 acts on S^3.

Exercise 2. The Möbius band.

The open Möbius band is the quotient

$$M = [0,1] \times (0,1) \,/\, ((0,y) \sim (1, 1-y) \text{ for all } y \in (0,1)).$$

Briefly explain why M is a smooth surface. Find (a homeomorphic copy of) M inside the real projective space $\mathbb{R}P^2$ and inside the Klein bottle K. The Möbius band $\overline{M}$, is obtained by replacing $(0,1)$ by $[0,1]$ above. Show[1] that the boundary of $\overline{M}$ is homeomorphic to S^1. Show that $\mathbb{R}P^2 = \text{(closed disc)} \cup \overline{M}$ glued along the circular boundary, and $K = \overline{M} \cup \overline{M}$.

Exercise 3. Riemann surfaces arising from polynomial equations.

Briefly explain a natural way to make the sets

(1) $S_1 = \{(z,w) \in \mathbb{C}^2 : w^2 = (z-1)(z-2)\} \cup \{+\infty\} \cup \{-\infty\}$
(2) $S_2 = \{(z,w) \in \mathbb{C}^2 : w^2 = (z-1)(z-2)(z-3)\} \cup \{\infty\}$

into Riemann surfaces. Find homeomorphisms $S_1 \cong$ sphere, $S_2 \cong$ torus. *(Hints in footnote[2])*

Exercise 4. The Klein bottle as a quotient of $\mathbb{R}^2$.

Consider the quotient

$$S = \mathbb{R}^2/G$$

where $G = \mathbb{Z}^2$ acts by $(n,m) \bullet (x,y) = ((-1)^m x + n, y + m)$ on $\mathbb{R}^2$, where $n, m \in \mathbb{Z}$. Briefly explain why S is a smooth surface. Show that S is homeomorphic to the Klein bottle.

Exercise 5. The space of lines in $\mathbb{R}^2$.

Let S be the set of all straight lines in $\mathbb{R}^2$ (not necessarily through 0). Show that there is a natural[3] way to make S into a topological surface. Show that S is homeomorphic to the open Möbius band M.

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[1] *Hint. Try cutting out a clever disc from $\mathbb{R}P^2$.*

[2] Hints. It helps if you first ask yourself what local holomorphic coordinate you would use at solutions (z,w) of $w^2 = z$ (recall from lecture notes the discussion of the square root $z^{1/2}$). Then try to build the solution set S_1 by gluing two cut-domains: two copies of $\mathbb{C}$ cut from 1 to 2. Just like for $\operatorname{Log} z$ in lectures, each subset you cut gives rise to *two* copies of that subset in the Riemann surface. In order to be able to draw the Riemann surface inside $\mathbb{R}^3$, it is convenient to reflect one of the cut-domains about the x-axis. Near infinity, try using the coordinates $X = \frac{1}{z}$ and $Y = \frac{w}{z}$ instead of z, w, and ask yourself what happens for $X = 0$ (corresponding to "$z = \infty$"). For S_2 you will need a second cut, from 3 to ∞, and try instead $Y = \frac{w}{z^2}$.

[3] *Hint. For example, lines which are not vertical can be parametrized by 2 numbers: the angle $\theta \in (-\pi/2, \pi/2)$ which tells you how much the line is tilted, and $r \in \mathbb{R}$ which is the signed distance of the line from the origin $0 \in \mathbb{R}^2$ (using the $+$ sign if the line passes above 0, and the $-$ sign if it passes below 0).*