B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 3

Comments and corrections are welcome: ritter@maths.ox.ac.uk

Exercise 1. Bump functions and embedding the Klein bottle in $\mathbb{R}^4$.

Recall from analysis that $\alpha(x) = e^{-1/x^2}$ is a function $\mathbb{R} \to \mathbb{R}$ (defining it to be zero at $x = 0$) which is infinitely differentiable, so smooth, but all the derivatives at $x = 0$ vanish! (so the Taylor series at 0 is useless)

$\diamond$ Sketch the following functions (no need to justify your sketches):

(1) $\beta(x) = \alpha(x)$ for $x > 0$, and $\beta(x) = 0$ for $x \leq 0$.
(2) $\gamma(x) = \beta(x-a) \cdot \beta(b-x)$, where $0 < a < b$.
(3) $\delta(x) = \int_x^b \gamma(t)\, dt / \int_a^b \gamma(t)\, dt$.

$\diamond$ The function $\varepsilon : \mathbb{R}^n \to \mathbb{R}$, $\varepsilon(x) = \delta(\|x\|)$ is called a *bump function compactly supported on the disc* $\|x\| \leq b$. Check that $\varepsilon(\mathbb{R}^n) \subset [0,1]$, that $\varepsilon = 1$ on $\|x\| \leq a$ and $\varepsilon = 0$ on $\|x\| \geq b$.

$\diamond$ A figure 8 loop can be obtained as the image of a continuous map $f : S^1 \to \mathbb{R}^2$ which is injective except at $\pm 1 \in S^1 \subset \mathbb{C}$ where $f(+1) = f(-1)$ (so f is not an embedding). Using a bump function, show that the figure 8 loop can be continuously embedded into $\mathbb{R}^3$.

$\diamond$ Show that the Klein bottle K can be smoothly embedded in $\mathbb{R}^4$.

$\diamond$ **Optional harder question:** Show that all compact surfaces can be embedded in $\mathbb{R}^4$.

Exercise 2. Holomorphic maps between Riemann surfaces.

Using the local form of a holomorphic map between Riemann surfaces, deduce:
Open mapping theorem: any holomorphic map $f : R \to S$ between Riemann surfaces, with R connected, is either constant or an open map, meaning f(any open set) is open.[1]

Deduce the following, for $f : R \to S$ holomorphic, R, S Riemann surfaces:

(1) If f is non-constant, R compact connected, then $f(R) \subset S$ is a connected component.
(2) If f is non-constant, R, S both compact connected, then f is surjective: $f(R) = S$.
(3) If R is compact connected, S non-compact connected, then f is constant.
(4) A holomorphic map $S \to \mathbb{C}$ on a compact connected Riemann surface is constant.
(5) Fundamental theorem of algebra: non-constant complex polynomials have a root.

Exercise 3. Implicit function theorem.

Consider $R = \{(z, w) \in \mathbb{C}^2 : w^3 = z^3 - z\}$. Use the implicit function theorem to check that R is a Riemann surface. Now consider the projection

$$\pi : R \to \mathbb{C},\ \pi(z, w) = z.$$

Find the branch points of π. Find the valency $v_\pi(p)$ at the ramification points.

Next, we seek how many points are "missing" at infinity. Write $z^3 - z = z^3(1 - z^{-2})$ for large $|z|$, and briefly explain that there are three holomorphic solution functions to $w^3 = z^3 - z$. Deduce that $\pi^{-1}(\{z \in \mathbb{C} : |z| > 100\})$ is biholomorphic to three punctured discs.

Compute the Euler characteristic of R using the Riemann-Hurwitz formula. Deduce that R is homeomorphic to a torus with three points removed.

Please turn over.

Date: This version of the notes was created on December 8, 2014.

[1] *Hint. Notice that to show a map is open, it's enough to show that for each p, there are some nice arbitrarily small open neighbourhoods of p which map to open sets.*

Exercise 4. Tangent space and orientability of Riemann surfaces.

For an open set $U \subset \mathbb{R}^2$, the tangent space at $p \in U$ is defined as the vector space $T_pU = \mathbb{R}^2$. You can visualise $v \in T_pU$ as an arrow pointing out of $p \in U$ in the plane. You should view v as the velocity vector $c'(0)$ of a smooth curve $c : (-\varepsilon, \varepsilon) \to U$ through $c(0) = p$. There are many different smooth curves c with $c'(0) = v$, $c(0) = p$, so v is an equivalence class of such.

$\diamond$ Write down the formula for a very simple curve $c : \mathbb{R} \to \mathbb{R}^2$ with $c(0) = p$, $c'(0) = v$. Deduce a simple formula for a curve which represents the vectors $v_1 + v_2$ and λv for $\lambda \in \mathbb{R}$. What does your formula become in components when v is a standard basis vector $e_1, e_2 \in \mathbb{R}^2$?

Let S be an (abstract) smooth surface. Given $p \in S$, we define T_pS as the collection of equivalence classes of smooth curves $c = c(t) : \mathbb{R} \to S$, defined for small $|t|$, with $c(0) = p$. Here $c \sim \widetilde{c}$ are equivalent if in some local coordinates $c'(0) = \widetilde{c}'(0)$.

$\diamond$ Using curves, there is an obvious way to identify $T_uU \equiv T_pS$ when you are given a local parametrization $F : \mathbb{R}^2 \supset U \to S$ with $p = F(u) \in F(U)$. Show that the composite $T_uU \equiv T_pS \equiv T_{\widetilde{u}}\widetilde{U}$, for overlapping parametrizations $F, \widetilde{F}$, becomes the map $D_u(\widetilde{F}^{-1} \circ F) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ (left-multiplication by the matrix of partial derivatives of the transition).

$\diamond$ Show that for S a Riemann surface, T_pS is a vector space with a well-defined linear isomorphism $J : T_pS \to T_pS$ with $J^2 = -\mathrm{Id}$ which in local coordinates is multiplication by i.

$\diamond$ Consider the open Möbius band M from Exercise sheet 1, and consider the equator $f : [0,1] \to M$, $t \mapsto (t, \frac{1}{2})$. Show that M cannot be a Riemann surface, by considering what J does to the tangent vector $f'(t) \in T_{f(t)}M$ as t varies from 0 to 1.

$\diamond$ Show that for a Riemann surface S, for any vector $v \neq 0 \in T_pS$, the basis v, Jv is a right-handed basis in any local parametrization.

Cultural remark. *For a smooth map $f : S_1 \to S_2$ of surfaces you can now define the derivative map by*

$$D_pf \cdot [\textit{curve } c(t)] = [\textit{curve } f \circ c(t)],$$

and you can check that if you write f in local coordinates, this corresponds to the matrix of partial derivatives of f at p acting by left-multiplication on $c'(0) = v \in \mathbb{R}^2$. So

$$D_pf : T_pS_1 \to T_{f(p)}S_2$$

is a linear map between the tangent spaces. Example: $e_1, e_2 \in \mathbb{R}^2$ map to $\partial_x f, \partial_y f \in \mathbb{R}^2$.

One can also think of vectors as differential operators acting on smooth functions: for example, if $h : S \to \mathbb{R}$ is smooth, then locally $e_1 \cdot h$ means the partial derivative $\partial_x h \in \mathbb{R}$.