B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 4

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Exercise 1. Riemann-Hurwitz formula.
In the following, all spaces are compact connected Riemann surfaces, and all maps are holomorphic maps. Deduce from the Riemann-Hurwitz formula that:

(1) if $f : R \to S$ is not constant, then the genus $g(R) \geq g(S)$.
(2) if $f : \mathbb{C}P^1 \to S$ is not constant, then S is homeomorphic to a sphere.
(3) if $f : R \to S$ has degree 1 then f is a biholomorphism.
(4) if R admits a meromorphic function with only one pole of order 1, then $R \cong \mathbb{C}P^1$.

Exercise 2. Meromorphic functions on Riemann surfaces.
Show that a map $f : S \to \mathbb{C}P^1$ is meromorphic if and only if locally f is expressible as a quotient of holomorphic functions (where the denominator is not identically zero).

Show that if f, g are two meromorphic functions on a compact connected Riemann surface having the same zeros and the same poles (including multiplicities) then $f = \text{constant} \cdot g$.

By comparing Taylor series of $\wp, \wp'$ near ramification points, deduce by the previous part (by viewing the two sides of the equation below as meromorphic functions) that:

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

where $e_1 = \wp(\frac{1}{2}\omega_1)$, $e_2 = \wp(\frac{1}{2}\omega_2)$, $e_3 = \wp(\frac{1}{2}(\omega_1 + \omega_2))$, $\infty = \wp(0)$ are the branch points of $\wp$.

Exercise 3. Elliptic curves and the Weierstrass $\wp$-function.
The goal is to prove that the following is a biholomorphism:

$$\begin{array}{rcl} \mathbb{C}/\Lambda & \to & S = \{(Z, W) \in \mathbb{C}^2 : W^2 = 4(Z - e_1)(Z - e_2)(Z - e_3)\} \cup \{\infty\} \\ z & \mapsto & (\wp(z), \wp'(z)) \end{array}$$

where on the right we compactify as done in Exercise Sheet 1. *Here is a checklist/hints:*

(1) Explain why e_1, e_2, e_3 are distinct,
(2) Show S is a Riemann surface. In particular, what is the local holomorphic coordinate?
(3) Explain why the map is well-defined,
(4) Show that the map is holomorphic (do this carefully, locally),
(5) For very general reasons, explain why the map has to be surjective,
(6) Show that the degree of the map is 1, and use Exercise 1.

Exercise 4. Covering maps.
Find a non-constant holomorphic map from a genus 3 surface to a genus 2 surface with no branch points.

(Hint. What degree must the map have? Start by finding a 2-to-1 map $T^2 \to T^2$, where $T^2 = S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$)

Cultural Remark. *A non-constant holomorphic map $f : R \to S$ between compact connected Riemann surfaces is a* covering map *in the sense that each small enough open set in S is covered via f by a disjoint union of copies of it in R (indeed $\deg(f)$ copies). Think of it as locally looking like a "stack of pancakes" over the "plate" in S. When there are branch points, $f : R \to S$ is called a* ramified covering map*: it fails to be a covering at ramification points.*

Date: This version of the notes was created on November 6, 2014.