B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 5

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Exercise 1. Conformal and area-preserving parametrizations.

In this exercise, the first fundamental form has local matrix $A = \left(\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\right)$.

Show that a local parametrization is conformal (i.e. angle-preserving) $\Leftrightarrow$ $f = 0$, $e = g$

Show that a local parametrization is area preserving $\Leftrightarrow$ $\det \left(\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\right) = 1$

Show that the stereographic projection is conformal, by considering the parametrization given by the inverse of the stereographic projection:

$$\mathbb{C} \to S^2 \setminus (\text{North Pole}),\ (x, y) \mapsto \frac{1}{1 + x^2 + y^2}(2x, 2y, -1 + x^2 + y^2),$$

and show that the fundamental form for the sphere in these local coordinates is

$$I_F = \frac{4(dx^2 + dy^2)}{(1 + x^2 + y^2)^2}.$$

Cultural remark: That fundamental form on the sphere is called the *chordal metric*. It gives rise to another example of non-Euclidean geometry called *elliptic geometry.*

Exercise 2. Tangential derivatives and Christoffel symbols.

A vector field *is a smooth family of tangent vectors* $v \in TS$, *so locally*

$$v(x, y) = a(x, y)\partial_x F + b(x, y)\partial_y F \in T_{F(x,y)}S,$$

for some smooth functions a, b. *The derivatives of* v *in* x, y *may not lie in* TS, *they lie in* $\mathbb{R}^3 = T_{F(x,y)}S \oplus \mathbb{R}n(x, y)$ *where* n *is the unit normal (Gauss map). If we subtract the normal part, we obtain the* tangential derivative*:*

$$\nabla_x v = \partial_x v - (n \cdot \partial_x v)\, n \qquad \nabla_y v = \partial_y v - (n \cdot \partial_y v)\, n.$$

The symbol ∇ *is called* nabla, *and the operator* ∇ *is called a* connection *for the surface* S.

Using that $v \in TS$ is orthogonal to n, show that $-n \cdot \partial_x v = \partial_x n \cdot v$. Show that ∇ is *compatible* with the Riemannian metric I (the first fundamental form):

$$\partial_x\, I(v, w) = I(\nabla_x v, w) + I(v, \nabla_x w),$$

for any vector fields v, w.

We introduce some helpful notation: $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, $\nabla_1 = \nabla_x$, $\nabla_2 = \nabla_y$. *Abbreviate the basis of* TS *by*

$$X_1 = \partial_x F \qquad \text{and} \qquad X_2 = \partial_y F.$$

Abbreviate $X_{ij} = \partial_i \partial_j F$. *Writing* $\nabla_i X_j \in TS$ *in the basis* X_i, X_j, *yields coefficient functions* $\Gamma^k_{ij}(x, y)$, *called* Christoffel symbols*:*

$$\boxed{\nabla_i X_j = \sum_{k=1}^{2} \Gamma^k_{ij} X_k}$$

Verify the symmetry relations:

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

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Now abbreviate by II_{ij} the entries of the second fundamental form $I = \left(\begin{smallmatrix} L & M \\ M & N \end{smallmatrix}\right)$, defined by

$$\boxed{II_{ij} = n \cdot X_{ij}}$$

Show that the normal part of the derivatives of X_j determines the second fundamental form:[1]

$$X_{ij} - \nabla_i X_j = II_{ij} n.$$

Let's abbreviate the entries I_{ij} of the first fundamental form $I = \left(\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\right)$ by:

$$\boxed{g_{ij} = I(X_i, X_j)}$$

Notice that $g_{ij} = g_{ji}$. Now prove that

$$X_{ij} \cdot X_\ell = I(\nabla_i X_j, X_\ell) = \sum_{k=1}^{2} \Gamma^k_{ij} g_{k\ell}.$$

Using the earlier compatibility result, and the above equation, show that

$$\partial_i g_{j\ell} = \sum_{k=1}^{2} (\Gamma^k_{ij} g_{k\ell} + \Gamma^k_{i\ell} g_{kj}).$$

Writing g^{ij} for the entries[2] of the inverse matrix $I^{-1} = \left(\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\right)^{-1}$, deduce that

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{\ell=1}^{2} g^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}).$$

Deduce that the Christoffel symbols are determined just by I, and therefore they are invariant under isometries between surfaces.

Exercise 3. Nautical cartography: Mercator's projection.

Using the parametrization

$$F(\theta, \phi) = (\cos\theta \cos\phi, \sin\theta \cos\phi, \sin\phi)$$

of the unit sphere $S^2 \subset \mathbb{R}^3$, find the first fundamental form.

To draw maps of the Earth, one often uses Mercator's projection of the unit sphere minus the date line:

$$(X, Y) = (\theta, \log\tan(\frac{\phi}{2} + \frac{\pi}{4})) \in \mathbb{R}^2,$$

where (θ, ϕ) are the longitude and latitude coordinates on the Earth. What does the first fundamental form of the sphere become in the coordinates (X, Y) of the plane? Deduce that Mercator's projection is conformal but not area-preserving. There are pictures of this on Wikipedia: http://en.wikipedia.org/wiki/Mercator_projection

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[1] In case you've lost the plot, an example should clarify:

$$X_{11} = \partial_x \partial_x F = \Gamma^1_{11} \partial_x F + \Gamma^2_{11} \partial_y F + L\, n$$

where L is the $(1, 1)$-entry II_{11} of the second fundamental form II.

[2] *Hint.* $I^{-1} I = \mathrm{id}$ *becomes the orthonormality equation*

$$\sum_{j=1}^{2} g^{ij} g_{jk} = \delta^i_k$$

where δ^i_k equals 1 for $i = k$ and 0 for $i \neq k$.