B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 6

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Exercise 1. The curvatures for a torus in $\mathbb{R}^3$.

Recall the torus in $\mathbb{R}^3$ given by

$$T^2 = \{((a + b\cos\psi)\cos\theta,\ (a + b\cos\psi)\sin\theta,\ b\sin\psi) :\ \text{all } \theta, \psi \in [0, 2\pi]\}$$

where $a > b > 0$ are fixed constants. Calculate the first fundamental form, the second fundamental form, the principal directions, the principal curvatures, the mean curvature and the Gaussian curvature K.

In a picture, shade the regions where K is positive, zero, and negative.

Exercise 2. Riemann curvature, Ricci curvature, scalar curvature.

Recall in Exercise sheet 5 we defined the tangential derivative, which in the basis $X_1 = \partial_x F$, $X_2 = \partial_y F$ *defines the Christoffel symbols:*

$$\nabla_i X_j = \Gamma^k_{ij} X_k \tag{*}$$

where from now on we use the Einstein summation convention*: you sum over repeated indices (so above, we sum over k since it appears once as an upper index and once as a lower index).*

The Riemann curvature tensor R^m_{ijk} *measures how much the tangential derivatives fail to commute. It is defined by:*

$$\boxed{\begin{array}{rcl} R(X_i, X_j)X_k & = & \nabla_i \nabla_j X_k - \nabla_j \nabla_i X_k \\ & = & R^m_{ijk} X_m \end{array}} \tag{**}$$

Cultural Remark. *We only defined R on the basis* $X_i = \partial_i F$*. The general formula is*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where the Lie bracket $[X, Y]$ *is defined by:* $[\sum v^i X_i, \sum w^j X_j] = \sum v^i \partial_i(w^j)X_j - \sum w^j \partial_j(v^i)X_i$, *which measures how much the flow of two vector fields fails to commute, and one studies this in* **C3.5 Lie Groups** *and secretly in* **C2.1 Lie Algebras**. *In our case:* $[X_i, X_j] = \partial_i(1)X_j - \partial_j(1)X_i = 0$ *since* 1 *is a constant coefficient.*

It's often useful to dot the above with another basis vector X_ℓ*, which defines*

$$\boxed{R_{ijk\ell} = R(X_i, X_j)X_k \cdot X_\ell = I(R(X_i, X_j)X_k, X_\ell)}$$

You can pass from one to the other by the lowering/raising of indices using the Riemannian metric $g_{ij} = I_{ij} = X_i \cdot X_j$*. Explicitly* $R_{ijk\ell} = R^m_{ijk} g_{m\ell}$*, which you can undo by using the inverse matrix* g^{ij} *of* g_{ij} *which satifies* $g^{ij} g_{jk} = \delta^i_k$ *(summing over j).*

By substituting (*) into (**), show that R is determined completely by the Christoffel symbols and the Riemannian metric $g_{ij} = I_{ij}$ (the first fundamental form):

$$\boxed{\begin{array}{rcl} R^m_{ijk} & = & \partial_i \Gamma^m_{jk} - \partial_j \Gamma^m_{ik} + \Gamma^p_{jk} \Gamma^m_{ip} - \Gamma^p_{ik} \Gamma^m_{jp} \\ R_{ijk\ell} & = & (\partial_i \Gamma^m_{jk} - \partial_j \Gamma^m_{ik} + \Gamma^p_{jk} \Gamma^m_{ip} - \Gamma^p_{ik} \Gamma^m_{jp}) g_{m\ell}. \end{array}}$$

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Notice that by Exercise Sheet 5, the Γ_{ij}^k *are determined by* g_{ij}*, so* R *only depends on the Riemannian metric* g_{ij}*. Therefore* R *doesn't change under isometry, even if you pick a different isometric embedding of the surface into* $\mathbb{R}^3$*.*

Explain why $R_{ijk\ell}$ is anti-symmetric in the indices i, j.

Recall by Exercise Sheet 5 that

$$\partial_i I(v,w) = I(\nabla_i v, w) + I(v, \nabla_i w).$$

Since $I(v,w)$ is a smooth function, its partial derivatives commute: $\partial_j \partial_i I(v,w) = \partial_i \partial_j I(v,w)$. Deduce that $R_{ijk\ell}$ is anti-symmetric in k, ℓ.

Deduce that R is determined by just one value: R_{1212}.

Recall by Exercise Sheet 5 that

$$\partial_i X_j = \nabla_i X_j + II_{ij}\, n = \Gamma_{ij}^k X_k + II_{ij}\, n.$$

Since F is smooth, the partial derivatives commute: $\partial_2 \partial_1 \partial_1 F = \partial_1 \partial_1 \partial_2 F$, so $\partial_2 \partial_1 X_1 = \partial_1 \partial_1 X_2$. From this, and the above equation, deduce by brute force calculation that

$$0 = (\partial_2 \partial_1 X_1 - \partial_1 \partial_1 X_2) \cdot X_2 = R_{2112} - \det II_F.$$

Deduce, using $K = \frac{\det II_F}{\det I_F}$ (from lectures), that

$$\boxed{R_{1212} = -K \det I_F}$$

Finally, deduce Gauss' *Theorema Egregium*: the Gaussian curvature K only depends on the Riemannian metric, so it is the same for two isometric surfaces.

The Ricci curvature *is defined as the metric trace of* $R_{ijk\ell}$ *in the* j, ℓ *indices, explicitly:*

$$R_{ik} = R_{ijk\ell} g^{j\ell} = R_{ijk}^j$$

and the scalar curvature *is the metric trace of the Ricci curvature, explicitly:*

$$R = g^{ij} R_{ij}$$

as usual summing over repeated indices.

Show that for surfaces in $\mathbb{R}^3$,

$$\boxed{\mathrm{R}_{ij} = -K g_{ij} \quad \text{and} \quad R = -2K}$$

Cultural Remark. *The above ideas are very important, for example the Einstein field equations for general relativity are*

$$R_{ij} - \frac{1}{2} g_{ij} R + g_{ij} \Lambda = \frac{8\pi G}{c^4} T_{ij}$$

where the left-hand side encodes the geometry of the universe and the right-hand side encodes the physical properties of the universe. The symbols are: G *= Newton's gravitational constant,* c *= speed of light,* Λ *= cosmological constant,* T_{ij} *= stress-energy tensor (which measures the matter/energy content of spacetime). More of this in* **C7.5/C7.6 General Relativity***.*

In a vacuum, these equations become $R_{ij} = 0$ *when the cosmological constant is zero, and* $R_{ij} = \Lambda g_{ij}$ *(so a multiple of the metric) otherwise. Manifolds with a vanishing Ricci tensor are called* Ricci-flat manifolds*, and manifolds with a Ricci tensor proportional to the metric are called* Einstein manifolds*. They are objects of great interest nowadays in geometry.*