

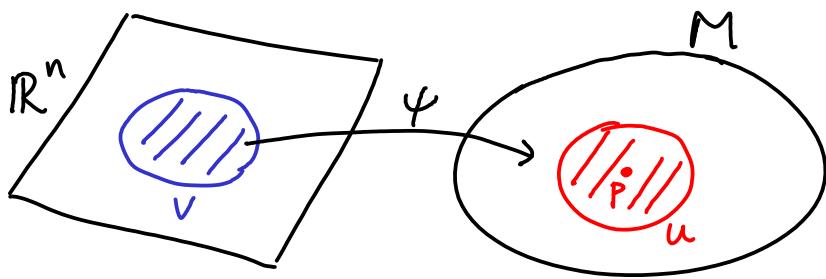
Let G be a group with operations

$$\begin{aligned}\mu: G \times G &\longrightarrow G & \mu(g, h) = gh \\ i: G &\longrightarrow G & i(g) = g^{-1}\end{aligned}$$

Def G is a Lie group if it is also a (smooth) manifold such that μ, i are smooth maps.

CRASH COURSE ON MANIFOLDS

A manifold $M = M^n$ of dimension n is a topological space which is locally parametrized by \mathbb{R}^n ,



A parametrization is a homeomorphism $\psi: V \rightarrow U$ (cts with cts inverse) from an open set $V \subset \mathbb{R}^n$ to an open set $U \subset M$.

Abbreviate this by

$$\psi: \mathbb{R}^n \dashrightarrow M$$

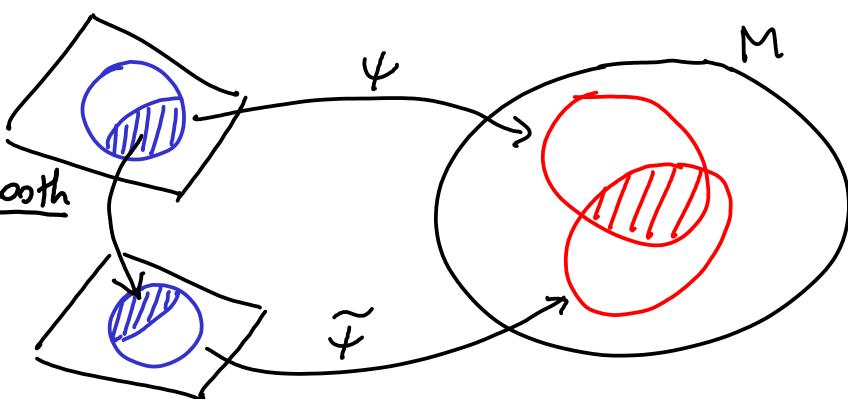
[Not standard notation - only this course]

such that on overlaps the parametrizations differ by smooth maps.

transition map

$$\tau = \tilde{\psi}^{-1} \circ \psi \text{ is smooth}$$

(all derivatives exist, as a map $\mathbb{R}^n \dashrightarrow \mathbb{R}^n$)

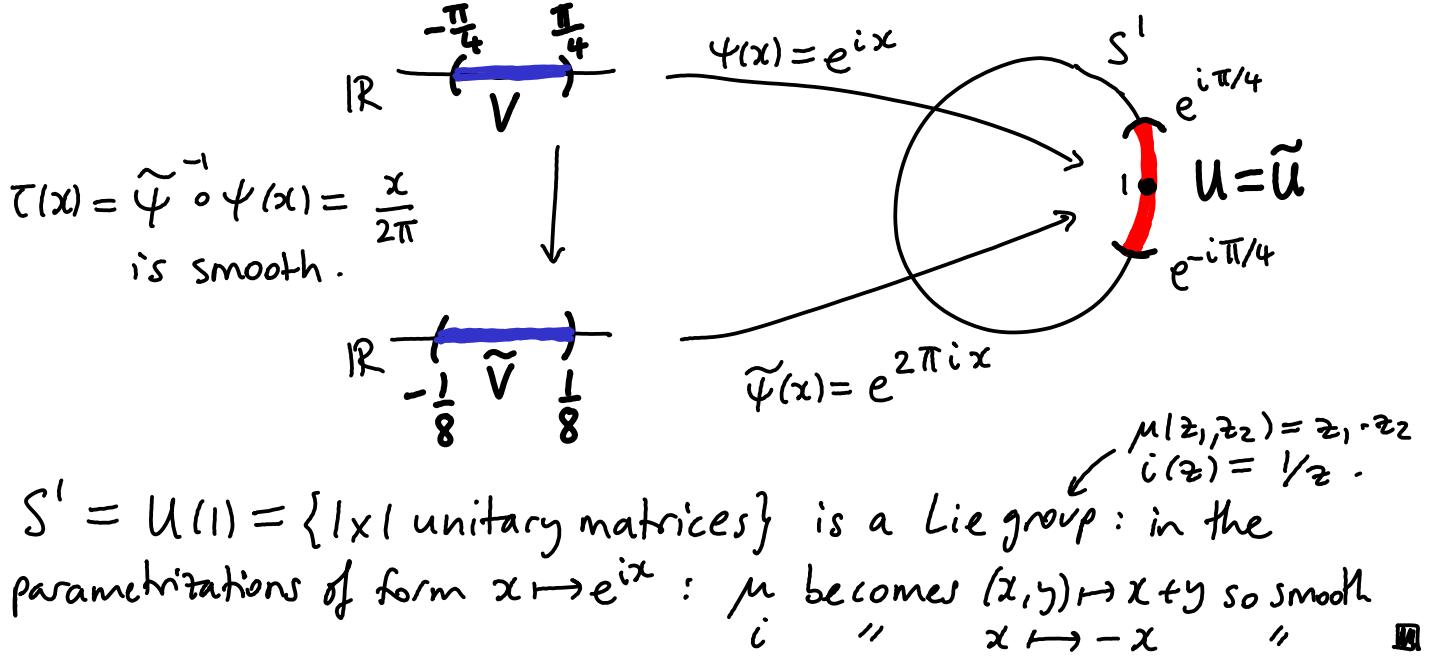


Rmks

- for a more precise definition, see the manifolds course.
- $\psi^{-1}: U \rightarrow V$ is called a chart

Def A parametrization $\psi: \mathbb{R}^n \dashrightarrow M$ defines local coordinates x_1, x_2, \dots, x_n on M
namely: $p \in U$ has coords $\psi^{-1}(p) = (x_1, \dots, x_n) \in \mathbb{R}^n$.

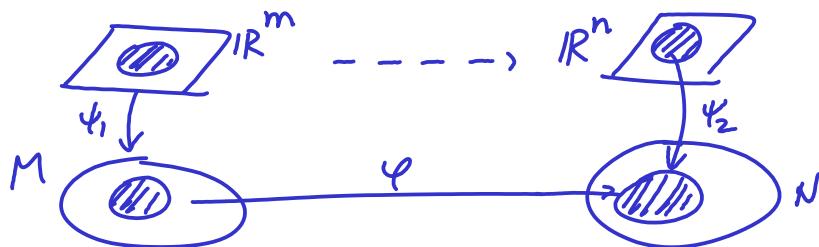
Example $M = S^1 = \text{circle} = \{z \in \mathbb{C} : |z|=1\}$



$S^1 = U(1) = \{1 \times 1 \text{ unitary matrices}\}$ is a Lie group: in the parametrizations of form $x \mapsto e^{ix}$: μ becomes $(x, y) \mapsto x + y$ so smooth
 i " " $x \mapsto -x$ " \blacksquare

SMOOTH MAPS

Def A continuous map $\varphi: M^m \rightarrow N^n$ of manifolds is smooth if locally in some (and hence all) parametrizations the map $\mathbb{R}^m \dashrightarrow \mathbb{R}^n$ is smooth.



locally: $\varphi(x) = (\varphi(x_1, \dots, x_n)) = (y_1(x), \dots, y_n(x))$
(so really mean) $\psi_2^{-1} \circ \varphi \circ \psi_1$ $x_i = \text{local coords near } p$ $y_i = \text{local coords near } \varphi(p)$

So: φ is smooth \Leftrightarrow the $y_i(x)$ are smooth functions of x .

VECTORS

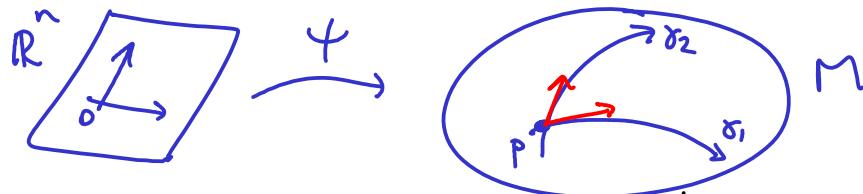
Def A (tangent) vector at $p \in M$ is an equivalence class $[\gamma]$ of smooth curves $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$, $\gamma(0) = p$ (some $\varepsilon > 0$)

equivalent $\gamma \sim \tilde{\gamma} \Leftrightarrow$ in some (hence all) parametrizations around p ,

$$\gamma'(0) = \tilde{\gamma}'(0) \in \mathbb{R}^n.$$

[locally $\gamma(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}^n \Rightarrow$ derivative $\gamma'(0) = (x'_1(0), \dots, x'_n(0)) \in \mathbb{R}^n]$

Example



$\psi: \mathbb{R}^n \dashrightarrow M$, $\psi(0) = p$, determines n obvious vectors at p

$$[\gamma_1(t) = \psi(t, 0, \dots, 0)] \text{ called } \frac{\partial}{\partial x_1}$$

$$[\gamma_2(t) = \psi(0, t, 0, \dots, 0)] \quad " \quad \frac{\partial}{\partial x_2}$$

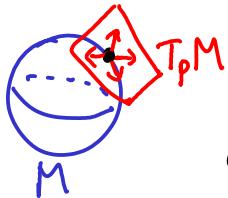
$$\dots$$

$$[\gamma_n(t) = \psi(0, \dots, 0, t)] \quad " \quad \frac{\partial}{\partial x_n}$$

Rmks

- Locally in \mathbb{R}^n those curves correspond to the standard basis vectors of \mathbb{R}^n : $\gamma_1'(0) = \frac{\partial}{\partial t}|_0 (t, 0, \dots, 0) = (1, 0, \dots, 0) \in \mathbb{R}^n$
 $\gamma_j'(0) = \frac{\partial}{\partial t}|_0 (0, \dots, 0, t, 0, \dots, 0) = (0, \dots, 0, 1, 0, \dots, 0)$
 - don't really need $\psi(0) = p$: if $\psi(x) = p$ just translate: $[\gamma_i(t) = \psi(x + (t, 0, \dots, 0))]$
 - can add/scale vectors: if $\psi(0) = p$, then just add/scale the curves in \mathbb{R}^n
- EXAMPLE : $2 \frac{\partial}{\partial x_1} + 4 \frac{\partial}{\partial x_3} = 2[\gamma_1] + 4[\gamma_3] = [\gamma(t) = \psi(2t, 0, 4t, 0, \dots, 0)]$

The tangent space $T_p M$ at $p \in M$ is the vector space of vectors at p



$$T_p M \quad \cong$$

$$\mathbb{R}^n$$

$$a_1 \frac{\partial}{\partial x_1} + \dots + a_n \frac{\partial}{\partial x_n} = [\underbrace{\psi(x + (a_1 t, \dots, a_n t))}_{\gamma(t) = \text{curve in } \mathbb{R}^n}] \mapsto \gamma'(0) = (a_1, \dots, a_n)$$

Rmk

- The isomorphism depends on the choice of ψ with $\psi(x) = p$. We will see later that changing ψ to another parametrization $\tilde{\psi}$ corresponds to multiplying $\gamma'(0) \in \mathbb{R}^n$ by the derivative $D\tau$ of the transition map $\tau = \tilde{\psi}^{-1} \circ \psi$.

Vectors act on functions

For $v = [\text{curve } \gamma(t)] \in T_p M$ $\Rightarrow v \cdot f = \frac{\partial}{\partial t} \Big|_0 f(\gamma(t)) \in \mathbb{R}$
 $f: M \rightarrow \mathbb{R}$ defined near p

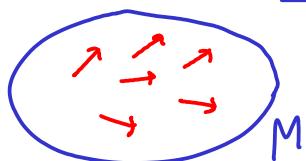
Locally it is just the obvious differentiation:

$$v = a_1 \frac{\partial}{\partial x_1} + \dots + a_n \frac{\partial}{\partial x_n} \quad v \cdot f = a_1 \frac{\partial f}{\partial x_1} + \dots + a_n \frac{\partial f}{\partial x_n} = \frac{\partial}{\partial t} \Big|_{t=0} f(a_1 t, \dots, a_n t)$$

where $\gamma(t) = (a_1 t, \dots, a_n t)$ is the local expression in parametrization ψ with $\psi(p)=0$

VECTOR FIELDS

Def A vector field is a map

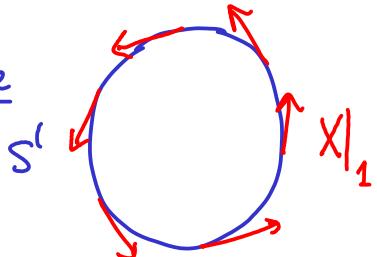


$$\begin{aligned} X: M &\longrightarrow TM = \bigsqcup_{p \in M} T_p M \\ p &\longmapsto X|_p \in T_p M \end{aligned}$$

such that locally $X|_x = a_1(x) \frac{\partial}{\partial x_1} + \dots + a_n(x) \frac{\partial}{\partial x_n}$ involves smooth functions $a_i(x) \in \mathbb{R}$.

[here $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i}|_x = [\psi(x + (0, \dots, 0, \underbrace{t}_\text{(position i)}, 0, \dots, 0))]$ is a vector at $x \in V$ so varies with x .]

Example



vector field $X = " \frac{\partial}{\partial \theta} "$ ($\theta = \text{angle} \in [0, 2\pi]$)

in a parametrization of type $\psi(x) = e^{ix}$
 have $X|_x = \frac{\partial}{\partial x} = [\text{curve } t \mapsto e^{i(x+t)}]$

Vector fields act on functions: $(X \cdot f)|_p = X|_p \cdot f$ so $X \cdot f$ is a function

Locally it's just differentiation

$$X \cdot f = a_1(x) \frac{\partial f}{\partial x_1} + \dots + a_n(x) \frac{\partial f}{\partial x_n} \quad \leftarrow \text{gives a new function of } x$$

Rmk • X is determined locally by differentiating the coordinate functions x_i : $X \cdot x_i = a_i(x)$

\Rightarrow A vector field is uniquely determined by how it acts on functions!

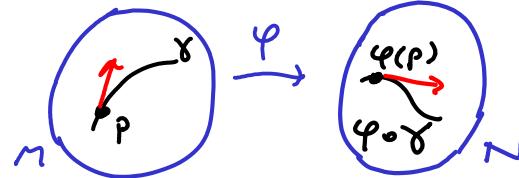
Rmk • When defining a concept locally, you always need to check that the choice of parametrization does not matter up to transition maps. So for $f: M \rightarrow \mathbb{R}$ we want $X \cdot f: M \rightarrow \mathbb{R}$ to be independent of the choices of ψ that locally define $X \cdot f$.
 (see Appendix if you care)

DERIVATIVE MAP

Def The derivative (or differential) of $\varphi: M \rightarrow N$ is

$$D\varphi: TM \rightarrow TN \quad (\text{in particular } D_p\varphi: T_p M \rightarrow T_{\varphi(p)} N)$$

$$D_p\varphi \cdot [\gamma] = [\varphi \circ \gamma]$$



CLAIM the derivative of φ is a linear map which is locally the matrix of partial derivatives of φ .

Proof Locally $\varphi(x_1, \dots, x_m) = (y_1(x), \dots, y_n(x))$ as a map $\mathbb{R}^m \dashrightarrow \mathbb{R}^n$

(we abusively write φ although it really is $\psi_2^{-1} \circ \varphi \circ \psi_1$, for parametrizations ψ_1 on M near p and ψ_2 on N near $\varphi(p)$).

$$\text{By definition, } D\varphi \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = D\varphi \cdot \frac{\partial}{\partial x_i} = D\varphi \cdot [\gamma_i(t) = [t, 0, \dots, 0]] = [\varphi \circ \gamma_i(t)] \\ = [(y_1(t, 0, \dots, 0), \dots, y_n(t, 0, \dots, 0))]$$

Locally (that is using the above isomorphism $T_p M \cong \mathbb{R}^m$, $[\varphi \circ \gamma] \mapsto \gamma'(0)$) we just need to differentiate the curve in t at time $t=0$:

$$\frac{d}{dt}|_{t=0}(y_1(t, 0, \dots, 0), \dots, y_n(t, 0, \dots, 0)) = \left(\frac{\partial y_1}{\partial x_1}, \dots, \frac{\partial y_n}{\partial x_1} \right) \in \mathbb{R}^n$$

(which is now written in the basis $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ of $T_{\varphi(p)} N \cong \mathbb{R}^n$)

$$\left(\text{so explicitly: } D_p\varphi \cdot \frac{\partial}{\partial x_i} = \frac{\partial y_1}{\partial x_i} \frac{\partial}{\partial y_1} + \dots + \frac{\partial y_n}{\partial x_i} \frac{\partial}{\partial y_n} \in T_{\varphi(p)} N \right)$$

Similarly $D_p\varphi \cdot \frac{\partial}{\partial x_j} = \left(\frac{\partial y_1}{\partial x_j}, \dots, \frac{\partial y_n}{\partial x_j} \right)$ and in general:

$$D_p\varphi \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = D_p\varphi \cdot \left(a_1 \frac{\partial}{\partial x_1} + \dots + a_n \frac{\partial}{\partial x_n} \right) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \dots & \dots & \frac{\partial y_n}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$$

(recall the columns of a matrix are the images of the standard basis) ■

Claim

CHAIN RULE

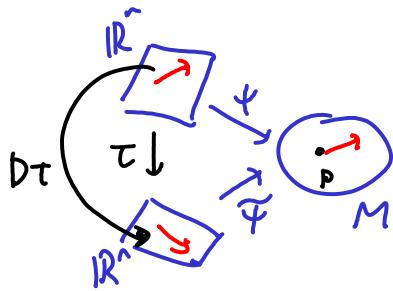
$$M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3 \Rightarrow D(\psi \circ \varphi) = D\psi \circ D\varphi$$

(locally it is
just the usual
chain rule)

Proof

$$D(\psi \circ \varphi) \cdot [\gamma] = [\psi \circ \varphi \circ \gamma] = D\psi \cdot [\varphi \circ \gamma] = D\psi \circ D\varphi \cdot [\gamma] ■$$

EXAMPLE The local expression of $\varphi = \text{identity} : M \rightarrow M$ if we use param. ψ near p on domain, and param. $\tilde{\psi}$ near $\varphi(p) = p$ on the image, is the transition map $\tau : \mathbb{R}^n \dashrightarrow \mathbb{R}^n$, $\tau = \tilde{\psi}^{-1} \circ \psi$.



Therefore $D\varphi = \text{identity}$ is locally $D\tau$.

Hence if $X^4, X^F \in \mathbb{R}^n$ are local expressions of the same vector X at $p \in M$ in parametrizations $\psi, \tilde{\psi}$ then

$$X^F = D\tau \cdot X^4$$

(see Appendix for more on this if you care)

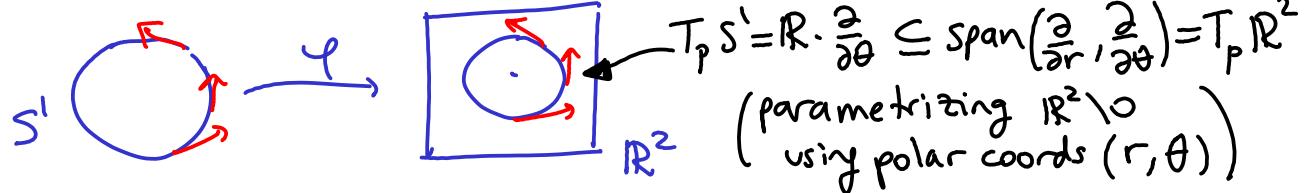
EXAMPLE $M = S^1$ vector field " $\frac{\partial}{\partial \theta}$ " is $X^4 = \frac{\partial}{\partial x}$ in param. $\psi(x) = e^{ix}$

However mathematicians often want to view S^1 as the quotient \mathbb{R}/\mathbb{Z} so they want to use $\tilde{\psi}(x) = e^{2\pi i x}$ (so that $\tilde{\psi}(\mathbb{Z}) = 1$). The local expression of " $\frac{\partial}{\partial \theta}$ " becomes: $X^F = D\tau \cdot X^4 = \frac{1}{2\pi} \cdot \frac{\partial}{\partial x}$ (since $\tau(x) = \frac{x}{2\pi}$, $D\tau = \frac{1}{2\pi} \cdot \text{Id}$)

Def $\varphi : M \rightarrow N$ is called an embedding if $\varphi : M \rightarrow \varphi(M)$ is a homeomorphism and $D_p \varphi : T_p M \rightarrow T_{\varphi(p)} N$ is injective $\forall p \in M$.

Rmks • Think of $\varphi(M) \subseteq N$ as an identical copy of M inside N
• $D_p \varphi$ injective $\Rightarrow D_p \varphi \cdot T_p M$ is a copy of $T_p M$ in $T_{\varphi(p)} N$ as a vector subspace

Example



EXAMPLE General linear group

$GL(n, \mathbb{R}) = \text{Lie group of invertible } n \times n \text{ real matrices} = \left\{ A : n \times n \text{ matrix with } \det A \neq 0 \right\}$
obvious parametrization near I :

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots \\ \vdots & & \\ a_{nn} & \dots & a_{nn} \end{pmatrix} \xleftarrow{\psi} (a_{11}, a_{12}, \dots, a_{21}, \dots, a_{nn}) \in \mathbb{R}^{n^2}$$

For any matrix $B = (b_{ij})$ have curve

$$\gamma(t) = A + tB = (a_{ij} + tb_{ij}) \xleftarrow{\text{still invertible for small } |t|} \text{so } A + tB \in GL(n, \mathbb{R})$$

$$\Rightarrow \text{vector } \frac{\partial}{\partial t} \Big|_0 \gamma(t) = B \in \text{Mat}_{n \times n}(\mathbb{R}) = \{n \times n \text{ real matrices}\}$$

$$\Rightarrow T_I GL(n, \mathbb{R}) = \text{Mat}_{n \times n}(\mathbb{R})$$

EXAMPLE orthogonal group

$O(n, \mathbb{R}) = \text{Lie group of orthogonal matrices} = \{A \in \text{Mat}_{n \times n} : A^T A = I\}$

Not easy to write down a parametrization near I

Note: the ψ for $GL(n, \mathbb{R})$ cannot work: ① the dimension is wrong and ② if you "wiggle" the a_{ij} then A may no longer be orthogonal!

(1) : $O(2, \mathbb{R}) = \{\text{rotations } \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}\} \sqcup \{\text{reflections } \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}\}$
 has dimension 1 obviously, since we give parametrizations in $\theta \in \mathbb{R}$
 Whereas $GL(2, \mathbb{R})$ has dimension $2^2 = 4 = \# \text{ entries.}$

Trick Consider the embedding $\psi: O(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R}), A \mapsto A$

and find $T_I O(n, \mathbb{R})$ as vector subspace of $T_I GL = \text{Mat}_{n \times n}$

For a curve $A(t) \in O(n, \mathbb{R}) \subseteq GL(n, \mathbb{R})$ through $A(0) = I$,

$$0 = \frac{d}{dt} \Big|_0 \underbrace{A(t)^T A(t)}_1 = \underbrace{A'(0)^T}_B \underbrace{A(0)}_I + \underbrace{A(0)^T}_I \underbrace{A'(0)}_B = B^T + B$$

$\Rightarrow T_I O(n, \mathbb{R}) \subseteq$ vector subspace of skew symmetric matrices
 $\uparrow \quad \{B \in \text{Mat}_{n \times n}(\mathbb{R}) : B^T + B = 0\}$

in fact, equality. One way to prove it is to check that $\dim O(n, \mathbb{R}) = \dim \{\text{skew } B\}$ since you are comparing vector spaces.

Question sheet: why is $O(n, \mathbb{R})$ a manifold? Need :

Implicit function theorem

Assume: $\varphi: M^m \rightarrow N^n$ smooth, $D_p \varphi: T_p M \rightarrow T_q N$ surjective
 for all $p \in \varphi^{-1}(q) = \{p \in M : \varphi(p) = q\}$.

- $\varphi^{-1}(q) \subseteq M$ is a submanifold ($\varphi^{-1}(q)$ is a manifold and the inclusion into M is an embedding)
- $\dim \varphi^{-1}(q) = m - n \quad \leftarrow \text{idea: } \varphi = q \text{ imposes } n = \dim N \text{ conditions (independent equations)}$
- $T_p(\varphi^{-1}(q)) = \text{Ker } D_p \varphi \quad \leftarrow \text{idea: if } \varphi = q \text{ is constant then } \varphi(\text{curve}) = \text{constant so zero vector.}$

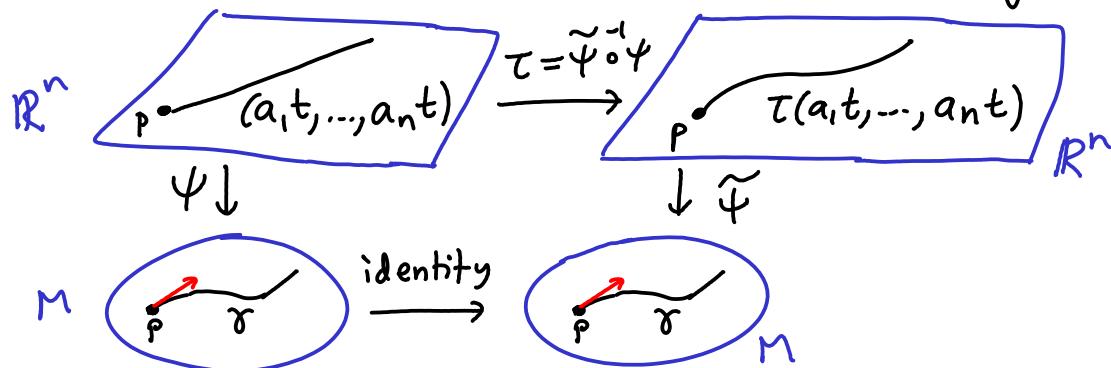
EXAMPLE: $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}, \varphi(x, y) = x^2 + y^2$. Take $q = 1$ then $\varphi^{-1}(1) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

= circle S^1 . $D\varphi = \text{matrix } (2x \ 2y) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is non-zero if $x^2 + y^2 = 1$ hence $D\varphi$ surjective.

Implicit fn thm $\Rightarrow S^1$ is mfd of $\dim = 2 - 1 = 1$ with $T_{(x,y)}S^1 = \text{Ker}((2x \ 2y) : \mathbb{R}^2 \rightarrow \mathbb{R})$
 as a vector subspace of $\mathbb{R}^2 \equiv T_{(x,y)}\mathbb{R}^2$. For example for $(x, y) = (1, 0)$ get $T_1 S^1 = \text{span}(\begin{pmatrix} 2 \\ 0 \end{pmatrix})$. $S^1 \bigcirc \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Appendix (NON-EXAMINABLE, also NOT IMPORTANT)

Question: how do vectors transform locally if change parametrization?



$$T_p M \cong \mathbb{R}^n \longrightarrow \mathbb{R}^n \cong T_p M$$

$$\sum a_i \frac{\partial}{\partial x_i} \cong (a_i)_{i=1, \dots, n} = \frac{\partial}{\partial t} |(a_i; t) \longmapsto \frac{\partial}{\partial t} | \tau \cdot (a_i; t) = \left(\sum_j \frac{\partial \tau_i}{\partial x_j} \cdot a_j \right) \cong \sum_{j,i} \frac{\partial \tau_i}{\partial x_j} a_j \frac{\partial}{\partial \tilde{x}_i}$$

↑ chain rule ↑ $\tau(x) \in \mathbb{R}^n$ has coordinates $\tau_i(x) \in \mathbb{R}$

\Rightarrow vectors transform by left-multiplication by the derivative $D\tau$ of the transition map $\tau: \mathbb{R}^n \dashrightarrow \mathbb{R}^n$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \longmapsto \begin{pmatrix} \frac{\partial \tau_1}{\partial x_1} & \frac{\partial \tau_1}{\partial x_2} & \dots & \frac{\partial \tau_1}{\partial x_n} \\ \frac{\partial \tau_2}{\partial x_1} & \dots & \dots & \frac{\partial \tau_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial \tau_n}{\partial x_1} & \dots & \dots & \frac{\partial \tau_n}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Question: is $X \cdot f = \sum a_i(x) \frac{\partial f}{\partial x_i} \in \mathbb{R}$ well-defined? (independent of choice of ψ)

Need more precise notation:

For $\psi: X^\psi = \sum a_i(x) \frac{\partial}{\partial x_i}, f^\psi(x) = f(\psi(x)), X^\psi \cdot f^\psi = \sum a_i(x) \frac{\partial f}{\partial x_i}$

For $\tilde{\psi}: X^{\tilde{\psi}} = D\tau \cdot X^\psi, f^{\tilde{\psi}}(y) = f(\tilde{\psi}(y)) = f^\psi(\tau^{-1}(y))$ $\tau^{-1} = \tilde{\psi}^{-1} \circ \tilde{\psi}$

$$X^{\tilde{\psi}} \cdot f^{\tilde{\psi}} = (D\tau \cdot X^\psi) \cdot (f^\psi \circ \tau^{-1}) \stackrel{\text{CHAIN RULE}}{=} D\tau \cdot X^\psi \cdot f^\psi \cdot D\tau^{-1} = X^\psi \cdot f^\psi$$

$\Rightarrow X^{\tilde{\psi}} \cdot f^{\tilde{\psi}} = X^\psi \cdot f^\psi$ agree at points of M . ■ check by writing it out with indices.