

## LECTURE 2

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### Examples of Lie groups

0) Finite groups (and discrete groups) : not so interesting since 0-dimensional manifolds (just a set of points)

lecture 1

$$\begin{cases} 1) \quad S^1 = U(1) \\ 2) \quad GL(n, \mathbb{R}) \end{cases}$$

3)  $O(n, \mathbb{R}) = \{\text{isometries of } \mathbb{R}^n \text{ which fix origin}\}$  include translations

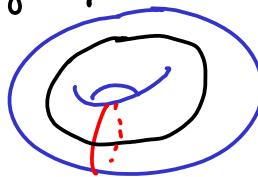
4)  $\text{Isom}(\mathbb{R}^n) = \{\text{all Euclidean isometries of } \mathbb{R}^n\}$   $x \mapsto x + c$ .

5)  $\mathbb{R}^n \leftarrow \mu(x, y) = x + y \text{ and } i(x) = -x$ .

6)  $G_1, G_2$  Lie groups  $\Rightarrow G_1 \times G_2$  Lie group

7)  $T^n = S^1 \times \dots \times S^1$  n-dimensional torus

$$T^2 = S^1 \times S^1$$



### The skew-field of quaternions

(non-commutative multiplication)

Def Quaternions  $\mathbb{H}$  : 3 equivalent definitions :

matrices	$\mathbb{C}$ - v.s. dim=2	$\mathbb{R}$ - v.s. dim=4
$\{h = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}\}$ $\mathbb{C} = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \right\} \subseteq \mathbb{H}$ $i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $j = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\{h = a + bj : a, b \in \mathbb{C}\}$ $\mathbb{C} = \{a + 0j\} \subseteq \mathbb{H}$ <u>how <math>j</math> multiplies</u> $j^2 = -1$ $jz = \bar{z}j \text{ for } z \in \mathbb{C}$	$\{h = x_1 + x_2i + x_3j + x_4k : x_i \in \mathbb{R}\}$ $\mathbb{C} = \{x_1 + x_2i\} \subseteq \mathbb{H}$ <u>quaternion relations</u> $i^2 = j^2 = k^2 = -1$ $ijk = -1$ (NOT COMMUTATIVE :) $ij = -ji = k$
$h^* := \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix}$ conjugate transpose $ h ^2 = \det =  a ^2 +  b ^2$	$h^* = \bar{a} - b j$ $ h ^2 = h^* h =  a ^2 +  b ^2$	$h^* = x_1 - x_2i - x_3j - x_4k$ $ h ^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$

scalar product  $\langle h_1, h_2 \rangle = h_1^* h_2 \in \mathbb{H}$  induces the norm  $|h| = \sqrt{\langle h, h \rangle}$

Rmk similar to  $\mathbb{C}$  : e.g.  $h \neq 0 \Rightarrow h^{-1} = h^*/|h|^2$

but careful about non-commutativity:  $(h_1 h_2)^* = \underline{\underline{h_2}} h_1^*$  ← what you expect when transpose matrices.

## More Examples of Lie groups

- 8)  $GL(1, \mathbb{H}) = \mathbb{H} \setminus \{0\}$  a "circle" in the world of quaternions  
 9)  $Sp(1) = \{ h \in \mathbb{H} : |h|=1 \} \subseteq \mathbb{H} \setminus \{0\}$  quaternion group  
 10) Generalize 5, 2, 3:

$\mathbb{R}^n$	$V$ vector space over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \text{ or } \mathbb{H}$
$GL(n, \mathbb{R})$	$\text{Aut}(V) = \{\mathbb{F}\text{-linear ssos } V \rightarrow V\}$ automorphism group
$O(n, \mathbb{R})$	$G = \{A \in \text{Aut } V : \langle Av, Aw \rangle = \langle v, w \rangle \forall v, w \in V\}$ $\star$ ↑ depending on a choice of scalar product $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$
11) In $\star$ : $V = \mathbb{F}^n$ with $\langle v, w \rangle = v^* w$	(* = conjugate transpose. Over $\mathbb{R}$ just transpose)

$\mathbb{F}$	Lie group $G \approx \{A \in GL(n, \mathbb{F}) : v^* A^* A w = v^* w \text{ all } v, w \in \mathbb{F}^n\}$
$\mathbb{R}$	orthogonal group $O(n) = O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : A^T A = I\}$
$\mathbb{C}$	unitary group $U(n) = \{A \in GL(n, \mathbb{C}) : A^\dagger A = I\}$
$\mathbb{H}$	symplectic group $Sp(n) = \{A \in GL(n, \mathbb{H}) : A^* A = I\}$

Ranks

- implicit function theorem  $\Rightarrow$  these are Lie groups,  
and  $T_I G = \{B \in \text{Mat}_{n \times n}(\mathbb{F}) : B^* + B = 0\}$
- fact:  $|Av| = |v| \quad \forall v \Leftrightarrow \langle Av, Aw \rangle = \langle v, w \rangle \quad \forall v, w \Leftrightarrow A^* A = I$   
norm  $|v| = \sqrt{\langle v, v \rangle}$

For  $O(n)$  geometrically this says linear maps which preserve lengths must preserve angles since  $\langle v, w \rangle = |v| \cdot |w| \cdot \cos(\text{angle between } v, w)$

- 12) special linear group  $SL(n, \mathbb{F}) = \{A \in GL(n, \mathbb{F}) : \det A = 1\} =$  volume preserving automorphisms  
 similarly:  $SO(n) \subseteq O(n)$  impose  $\det = 1$   
 $SU(n) \subseteq U(n)$

(Can't do it for  $Sp(n)$  since  $\det$  doesn't make sense since  $\mathbb{H}$  not commutative)

Q.Sheet 2:  $\det = 1$  imposes  $\text{trace}(B) = 0$  for tangent vectors  $B \in T_I G \subseteq \text{Mat}_{n \times n}$

Example  $O(2) = SO(2) \sqcup \{\text{reflections}\}$   
 rotations (connected component of  $I$ )  $\xrightarrow{\text{not a group}} \{\text{it is a coset}\}$  (no  $I$ !)

$$O(2) \cong SO(2) \times \{\pm 1\} \text{ via } A \mapsto (\det A) \cdot A, \det A$$

13) Def  $G_0 = (\text{connected component of } 1 \text{ in } G)$  ↪ Lie group

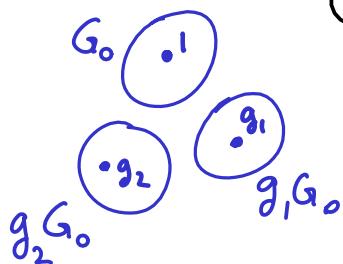
Lemma

①  $G_0$  is a Lie group

② the cosets of  $G_0$  in  $G$  are the connected components of  $G$  and they give an open cover  $G = \bigsqcup gG_0$

③  $G_0$  is a normal subgroup of  $G$   $gG_0 \in G/G_0 = \{\text{cosets}\}$

④ The quotient group  $G/G_0$  is a Lie group (discrete group so 0-manifold)



EXAMPLE  $G = O(n) \Rightarrow G_0 = SO(n)$  and  $G/G_0 \cong \{\pm 1\}$  (group with 2 elements)

Rmk's

- For subgroups  $H \leq G$  the space of cosets  $G/H$  may not be group.

$H \leq G$  called normal subgroup if  $hHh^{-1} \subseteq H \quad \forall h \in G$

This condition ensures that  $G/H$  is a group using  $g_1H \cdot g_2H = g_1g_2H$

- In general  $G \cong G_0 \times \frac{G}{G_0}$  is false! (see Appendix if you care)

Proof of Lemma

$\phi_g : G \rightarrow G, h \mapsto gh$  is a homeomorphism (inverse is  $\phi_{g^{-1}}$ )

(continuous - indeed smooth - since group multiplication is smooth)

$\Rightarrow \phi_g$  sends connected components to connected components

$\Rightarrow gG_0$  are connected components

Recall: any topological space = disjoint union of its connected components

Hence:  $g \in G_0 \Rightarrow g \in G_0 \cap gG_0 \Rightarrow G_0 = gG_0$  (since  $G_0 \cap gG_0 \neq \emptyset$ )  
 $\Rightarrow g^{-1}G_0 = G_0$

Therefore can restrict multiplication and inversion to  $G_0$ , proving ①

If  $C$  is a connected component and  $g \in C$ , then  $g \in C \cap gG_0 \neq \emptyset$ , so  $C = gG_0$   
 $\Rightarrow$  ② follows (using general fact: connected components of a manifold are always open sets)

For ③ use homeomorphism  $G \rightarrow G, h \mapsto ghg^{-1}$  (inverse  $h \mapsto g^{-1}hg$ )  
 $G_0$  connected component  $\Rightarrow gG_0g^{-1}$  connected component  
but  $1 \in G_0 \cap gG_0g^{-1} \neq \emptyset$  so  $G_0 = gG_0g^{-1}$ .

④ follows (not much content in ④ since silly manifold:  $\dim = 0$ ) ■

# TOPOLOGICAL PROPERTIES

## COMPACTNESS

recall compact means open covers always have finite subcovers.

- useful trick: 1) first embed  $G \subseteq \mathbb{R}^m$  (some large  $m$ )  
 2) then use Heine-Borel theorem for  $\mathbb{R}^m$ :

$$\text{compact} \Leftrightarrow \text{closed \& bounded}$$

that is: • check limits stay in  $G$

- $\mathbb{R}^m$ -norm on  $G \subseteq \mathbb{R}^m$  is bounded

- EXAMPLE  $S^1 \subseteq \mathbb{R}^2$  • if  $z_n \rightarrow z$  with  $|z_n|=1$  then  $|z|=1$  so  $z \in S^1$   
 •  $|z| \leq 1$  on  $S^1$  (since  $|z|=1 \forall z \in S^1$ )

## CONNECTEDNESS

- facts • manifolds are metric spaces (since can always embed  $M \subseteq \mathbb{R}^{\text{huge}}$ )  
 • for manifolds a subset is a connected component  $\Leftrightarrow$  open and closed  
 • for manifolds: connected  $\Leftrightarrow$  path-connected

## EXAMPLE

$$GL(1, \mathbb{R}) = \mathbb{R} \setminus \{0\} = \mathbb{R}^+ \sqcup \mathbb{R}^- \text{ not connected}$$

$$GL(1, \mathbb{C}) = \mathbb{C} \setminus \{0\} \text{ connected since path-connected.}$$

← DON'T NEED TO MEMORIZE THIS!

## Some Topological facts (some are tricky to prove)

$G$	connected ( $G=G_0$ )	# Connected components if #1	compact
$\mathbb{R}^n$ $T^n$	✓		X ✓ ( $n \geq 1$ )
$GL(n, \mathbb{R})$	X		X
$SL(n, \mathbb{R})$	✓		X ( $n \geq 2$ )
$O(n)$	X		✓
$SO(n)$	✓	$2 < \frac{\det}{\det} > 0$	✓
$GL(n, \mathbb{C})$			X
$SL(n, \mathbb{C})$			X ( $n \geq 2$ )
$U(n)$			✓
$SU(n)$	✓		✓
$GL(n, \mathbb{H})$ $Sp(n)$	✓		X ✓ ( $n \geq 2$ )

# WHY ARE LIE GROUPS SUCH SPECIAL MANIFOLDS?

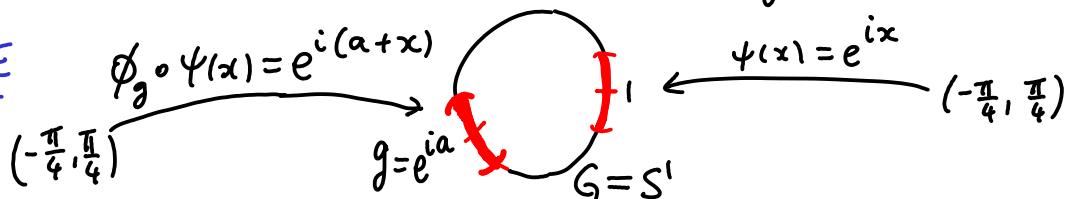
They have natural diffeomorphisms: (smooth with smooth inverse)  
NOTE:  $\phi_g^{-1} = \phi_{g^{-1}}$

$\phi_g: G \rightarrow G, h \mapsto gh$	left-translation by $g$
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Amazing consequences:

- ① Once you pick a parametrization near 1, say  $\psi: U \xrightarrow{\subset \mathbb{R}^n} V \subseteq G$  with  $\psi(0) = 1$ , get a parametrization near any  $g \in G$ :  $\phi_g \circ \psi: U \rightarrow g \cdot V$  with  $\phi_g \circ \psi(0) = g$

EXAMPLE



- ② In these parametrizations,  $\phi_g$  is locally the identity map!

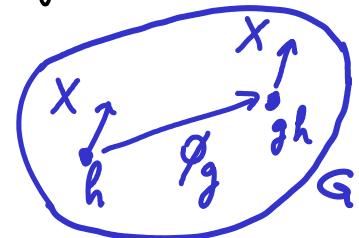
$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\text{Id=identity}} & \mathbb{R}^n \\ \psi \downarrow & & \downarrow \phi_g \circ \psi \\ G & \xrightarrow{\phi_g} & G \end{array}$$

Which vector fields  $X$  on  $G$  have exactly the same local expression near any  $g$  in parametrizations  $\phi_g \circ \psi$ ?

$$\begin{array}{c} \sum a_i(x) \frac{\partial}{\partial x_i}: D\text{Id} = \text{Id} \xrightarrow{\quad} \sum a_i(x) \frac{\partial}{\partial x_i}: D(\phi_g \circ \psi) = D\phi_g \circ D\psi \\ D\psi \downarrow \qquad \qquad \qquad \text{chain rule} \\ X|_{\psi(x)} \xrightarrow{D\phi_g} X|_{g \cdot \psi(x)} \end{array} \left. \begin{array}{l} \text{hence need } X|_{g \cdot \psi(x)} = D\phi_g \cdot X|_{\psi(x)} \end{array} \right\}$$

Def A vector field  $X$  on  $G$  is left-invariant if

$$D_h \phi_g \cdot X|_h = X|_{gh} \quad (\forall g, h \in G)$$



Def

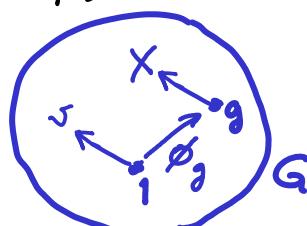
$$\boxed{\text{Lie } G = \{ \text{left-invariant vector fields on } G \}}$$

Rmk this is a vector space: adding/scaling vector fields preserves left-invariance because  $D\phi_g$  is linear.

Theorem There is a natural isomorphism of vector spaces:

$$\begin{array}{ccc} \text{Lie } G & \longrightarrow & T_1 G \\ X & \longmapsto & X|_1 \\ \left( \begin{array}{c} \text{vector field } X \\ X|_g = D_1 \phi_g \cdot v \end{array} \right) & \longleftarrow & v \end{array}$$

In particular  $\dim \text{Lie } G = \dim T_1 G = (\dim G \text{ as a manifold})$ .



basis  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$

## THE LIE BRACKET

For any vector fields  $X, Y$  on a manifold  $M$  there is a bracket operation  $[X, Y] =$  a new vector field, defined locally by

$$\left[ \sum a_i(x) \frac{\partial}{\partial x_i}, \sum b_i(x) \frac{\partial}{\partial x_i} \right] = \sum_j \sum_i \left( a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

EXAMPLE in  $\mathbb{R}^2$ :

$$[x^2 y^3 \frac{\partial}{\partial x}, x^4 y^5 \frac{\partial}{\partial y}] = x^2 y^3 \frac{\partial}{\partial x} (x^4 y^5) \frac{\partial}{\partial y} - x^4 y^5 \frac{\partial}{\partial y} (x^2 y^3) \frac{\partial}{\partial x} = -3x^6 y^7 \frac{\partial}{\partial x} + 4x^5 y^8 \frac{\partial}{\partial y}$$

Recall: vector fields act on functions, and this locally determines the vector field since  $X \cdot x_i = a_i(x)$  if  $X = \sum a_i(x) \frac{\partial}{\partial x_i}$  locally. For brackets:

$$[X, Y] \cdot f = X \cdot \underbrace{(Y \cdot f)}_{\text{new function}} - Y \cdot (X \cdot f)$$

↑ new function, then  $X$  differentiates it

IDEA:  $[X, Y]$  measures how badly  $X, Y$  fail to commute as differential operators

e.g.  $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$  says partial derivatives commute on smooth functions:

$$\frac{\partial^2}{\partial x_i \partial x_j} (f) = \frac{\partial^2}{\partial x_j \partial x_i} (f)$$

### PROPERTIES

- i)  $[\cdot, \cdot]$  is bilinear :  $\mathbb{R}$ -linear in each entry.
- ii) antisymmetric :  $[X, Y] = -[Y, X]$ , so  $[X, X] = 0$
- iii) Jacobi's identity :

exercise ↑

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

## LIE ALGEBRAS

Def A Lie algebra is a vector space  $V$  together with a bilinear antisymmetric map  $[\cdot, \cdot] : V \times V \rightarrow V$  satisfying the Jacobi identity.

EXAMPLES

- $V = \text{Mat}_{n \times n}(\mathbb{R})$ ,  $[B, C] = B C - C B$  (matrix multiplication)
- $V = \mathbb{R}^3$ ,  $[v, w] = v \times w$  cross-product
- Abelian Lie algebras : any vector space  $V$  with  $[\cdot, \cdot] = 0$

Theorem Lie  $G$  is a Lie algebra of dimension  $\dim G$

Pf Need show :  $X, Y$  left-invt  $\Rightarrow [X, Y]$  left-invt.

In parametrizations  $\phi_g \circ \psi$  the v.f.  $X$  has the same local expression near any  $g \in G$ , so does  $Y$ , hence so does  $[X, Y]$ , so  $[X, Y]$  is left-invt.

EXAMPLE  $gl(n, \mathbb{R}) = \text{Lie } GL(n, \mathbb{R})$

$$gl(n, \mathbb{R}) \equiv \text{Mat}_{n \times n}(\mathbb{R}) = T_I GL(n, \mathbb{R})$$

$$X = \sum_{i,j} X_{ij}(x) \frac{\partial}{\partial x_{ij}} \longleftrightarrow B = X|_I = \begin{pmatrix} X_{ij}(I) \\ i=1, \dots, n \\ j=1, \dots, n \end{pmatrix}$$

left-invt:

$$X|_x = D_x \phi_x \cdot X|_I = x \cdot B$$

$x \in GL(n, \mathbb{R})$

$x = (x_{ij}) \in \mathbb{R}^{n^2}$  entries

are local coords near I

$= \varphi(x) + t\varphi(v)$  since  $\varphi$  linear

(For linear maps  $\varphi$ , " $D\varphi = \varphi$ " since  $D_x \varphi \cdot v = \lim_{t \rightarrow 0} \frac{\varphi(x+tv) - \varphi(x)}{t} = \varphi(v)$ ).

$$[X, Y] = \sum_{i,j,k} \left( X_{ij} \frac{\partial Y_{ek}}{\partial x_{ij}} - Y_{ij} \frac{\partial X_{ek}}{\partial x_{ij}} \right) \frac{\partial}{\partial x_{ek}} \longleftrightarrow \sum (B_{ij} C_{jk} - C_{ij} B_{jk}) \frac{\partial}{\partial x_{ik}}$$

$$X|_x = x \cdot B \quad Y|_x = x \cdot C \quad (\text{so } Y_{ek} = \sum_j x_{ej} C_{jk} \text{ and } \frac{\partial Y_{ek}}{\partial x_{ij}} = C_{jk} \text{ for } l=i \text{ and zero otherwise})$$

Corollary  $gl(n, \mathbb{R}) \cong \text{Lie algebra Mat}_{n \times n}(\mathbb{R})$  with bracket  $[B, C] = BC - CB$

iso of Lie algebras  
 (= iso of vector spaces preserving bracket)  $\longleftrightarrow$  precise definition:  $\varphi: V \rightarrow W$  iso of v.s.  
 $\varphi([v_1, v_2]) = [\underbrace{\varphi v_1}_{\text{in } V}, \underbrace{\varphi v_2}_{\text{in } W}]$  all  $v_1, v_2 \in V$

Same calculation shows:

$$\text{being in } O(n) \text{ puts some constraints on the } X_{ij}(x) \text{ and on } x, \text{ but calculation still holds viewing } O(n) \subseteq GL(n)$$

$$O(n, \mathbb{R}) = \text{Lie } O(n, \mathbb{R}) \cong \{ B \in \text{Mat}_{n \times n}(\mathbb{R}) : B \text{ skew-symmetric} \}$$

(iso of Lie algebras using usual bracket for  $\text{Mat}_{n \times n}$ )

## APPENDIX (NON-EXAMINABLE - can ignore it)

### Remarks about $G_0$

• Not true in general that  $G \cong G_0 \times G/G_0$  as groups (not even  $G_0 \times G/G_0$ ) because there is no reason homomorphisms  $G \rightarrow G_0$  and  $G/G_0 \rightarrow G$  should exist. It is true that  $G \cong G_0 \times G/G_0$  as manifolds since you just pick some representatives  $g_i$ : then  $G_0 \times \{g_i G\} \rightarrow G$  but can't make choices consistently with the group structure.

Later in course we prove  $G \cong G_0 \times G/G_0$  works for abelian  $G$ .

$$(g, g_i G) \mapsto g \cdot g_i$$

• There may be bigger subgroups than  $G_0$ : for  $G = S^1 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ :  $G_0 = S^1 \times \{1\} \times \{1\} \subseteq S^1 \times \mathbb{Z}_2 \times \{1\} \subseteq G$ . But of course  $G_0$  is the largest connected subgroup of  $G$ .

ABOUT  $[X, Y]$  Why is  $[X, Y]$  a well-defined global vector field?

Answer: If for any function  $f$  defined near  $p \in M$  you define a new function  $Z \cdot f$  defined near  $p \in M$ , and you show the Leibniz rule holds  $Z \cdot (f_1 f_2) = (Z \cdot f_1) f_2 + f_1 (Z \cdot f_2)$  then  $Z$  is a vector field on  $M$  (try proving this using the Appendix of lecture 1).

We defined  $[X, Y] \cdot f$  and we defined  $[X, Y]$  in local coordinates: the two definitions agree when compute  $[X, Y] \cdot f$  locally, and the local definition clearly satisfies Leibniz. ■