

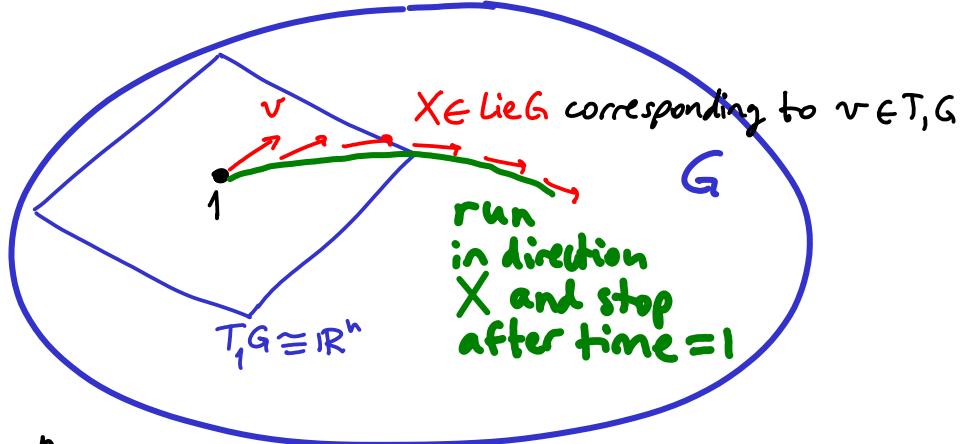
## LECTURE 3

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C3.5 LIE GROUPS, HT2015, Oxford.

LAST TIME: Parametrization  $\gamma$  near  $1 \in G \rightsquigarrow$  parametrizations  $\phi_g \circ \gamma$  near any  $g$

AIM: Find the best parametrization  $\gamma$  near 1

IDEA:



$$\psi: \mathbb{R}^n \stackrel{\text{choice of basis}}{\cong} T_1 G \stackrel{\text{left invariant}}{\cong} \text{Lie } G \longrightarrow G$$

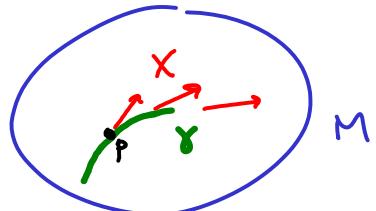
$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$

### FLows

Def For a vector field  $X$  on a manifold  $M$ , a flowline of  $X$  through  $p$  is a curve

$$\gamma: (-\varepsilon, \varepsilon) \longrightarrow M \quad (\varepsilon > 0)$$

$$\gamma(0) = p \quad \gamma'(t) = X|_{\gamma(t)}$$

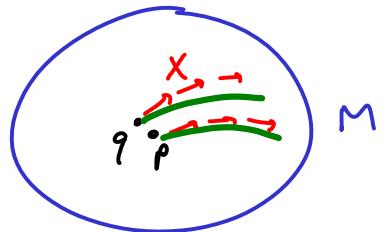


If let  $p$  vary: flow of  $X$  near  $p$  is

$$F: (-\varepsilon, \varepsilon) \times U \xrightarrow{\text{small open set around } p} M$$

$$F(0, q) = q \quad \frac{\partial F}{\partial t}|_{(t, q)} = X|_{F(t, q)}$$

$\Rightarrow F(\cdot, q)$  is flowline through  $q$ .



Rmks •  $\gamma'(t)$  is an abbreviation for  $D_t \gamma \cdot \frac{\partial}{\partial t}$ . Locally it's really  $\gamma'(t)$ .

• The equation is locally a 1<sup>st</sup> order diff. eqn. on  $\mathbb{R}^n$ :

$$x'(t) = f(x(t)) \quad x(0) = p \quad f: \mathbb{R}^n \xrightarrow{\text{smooth}}$$

ODE theory  $\Rightarrow$  •  $x$  exists for small  $\varepsilon > 0$

•  $x$  is unique given  $p$

•  $x$  depends smoothly on the initial condition  $p$

So for small  $\varepsilon$  and  $U$ ,  $\gamma$  and  $F$  exist, unique, smooth.

Example  $M = (-1, 1) \subseteq \mathbb{R}$ ,  $X = \frac{\partial}{\partial t}$  then  $F(q, t) = q + t$  only defined if  $q + t \in (-1, 1)$ .

Abbreviate  $F_t(g) = F(t, g)$

Lemma 1  $F_s(F_t(g)) = F_{s+t}(g)$  (where defined)

Pf for  $s=0$ : LHS = RHS =  $F_t(g)$

$$\frac{\partial}{\partial s} \Big|_{s=0} (\text{LHS}) = X|_{F_s(F_t(g))}, \quad \frac{\partial}{\partial s} \Big|_{s=0} (\text{RHS}) \stackrel{\text{chain rule}}{=} X|_{F_{s+t}(g)}$$

Now suppose  $M=G$  is a Lie group  
 $X \in \text{Lie } G$  left-invariant

Lemma 2  $\gamma$  flowline  $\Rightarrow g \cdot \gamma$  flowline

$$\text{If } \frac{\partial}{\partial t} \Big|_t (g \cdot \gamma(t)) = \frac{\partial}{\partial t} (\phi_g \circ \gamma(t)) = D\phi_g \cdot \gamma'(t) = D\phi_g \cdot X|_{\gamma(t)} = X|_{g \cdot \gamma(t)} \blacksquare$$

Start being sloppy:  
doing the derivative wherever it  
is relevant, here  $\gamma'(t)$  is a  
vector at  $\gamma(t)$  so we take  $D_{\gamma(t)} \phi_g$

Cor 1 flowlines of  $X \in \text{Lie } G$  are defined for all time, and flow defined everywhere

Pf  $\gamma: (-\varepsilon, \varepsilon) \rightarrow G \Rightarrow$  can extend on  $(0, \varepsilon + \frac{\varepsilon}{2})$  using  $\gamma(\frac{\varepsilon}{2}) \gamma(0)^{-1} \gamma(t - \frac{\varepsilon}{2})$

Explanation:  $t \mapsto \gamma(t - \text{constant})$  is a flowline: by chain rule  $\frac{\partial}{\partial t} \gamma(t - c) = \gamma'(t - c) = X|_{\gamma(t - c)}$  then apply Lemma 1 to flowline  $\gamma(t - \frac{\varepsilon}{2})$  with  $g = \gamma(\frac{\varepsilon}{2}) \gamma(0)^{-1}$ . So  $\delta(t) = \gamma(\frac{\varepsilon}{2}) \gamma(0)^{-1} \gamma(t - \frac{\varepsilon}{2})$  is a flowline for  $X$ . We constructed it so that  $\delta(\frac{\varepsilon}{2}) = \gamma(\frac{\varepsilon}{2}) \gamma(0)^{-1} \gamma(0) = \gamma(\frac{\varepsilon}{2})$ . But then  $\delta, \gamma$  are both flowlines for  $X$  and equal at  $t = \frac{\varepsilon}{2}$ , so they equal on overlap by uniqueness. So  $\delta$  extends  $\gamma$  beyond  $\varepsilon$ . Similarly can extend beyond  $-\varepsilon$ . Finally argue by contradiction: if  $(-\varepsilon, \varepsilon)$  was the largest interval where  $\gamma$  is defined, then we just showed that cannot be true since we can extend.

Key trick: let  $\gamma: \mathbb{R} \rightarrow G$  be the (unique) flowline of  $X$  with  $\gamma(0)=1$  then by Lemma 1 the flow of  $X$  is:  $F_t(g) = g \cdot \gamma(t)$  ■

Theorem Let  $\gamma$  be the flowline of  $X \in \text{Lie } G$  with  $\gamma(0)=1$ . Then

$$\gamma(s) \gamma(t) = \gamma(s+t) \quad \forall s, t \in \mathbb{R} \quad (\text{in particular: } \gamma(s)\gamma(t) = \gamma(t)\gamma(s))$$

So  $\gamma: \mathbb{R} \rightarrow G$  is a Lie group homomorphism. ← (hom of gps &)

Conversely, all Lie gp homs  $\gamma: \mathbb{R} \rightarrow G$  arise in this way for some  $X \in \text{Lie } G$ .

Pf. By Lemma 1 + Cor 1:  $F_s(F_t(1)) = F_{s+t}(1) = 1 \cdot \gamma(s+t)$   
 $\Downarrow F_s(1 \cdot \gamma(t)) = 1 \cdot \gamma(t) \cdot \gamma(s)$

Conversely, if  $\gamma$  homom  $\mathbb{R} \rightarrow G$  then  $\gamma(0)=1$ , and we claim  $F(t, g) = g \cdot \gamma(t)$  is flow for  $\frac{\partial}{\partial t} |_{t=0} F(t, g) = D\phi_g \cdot \gamma'(0) = X|_g$  left-inv v.f. determined by  $\gamma'(0) \in T_g G$ .

proof:  $F_{s+t}(g) = g \cdot \gamma(s+t) = g \cdot \gamma(t+s) = g \cdot \gamma(t) \cdot \gamma(s) = F_s(g \cdot \gamma(t)) = F_s F_t(g)$

and in general  $F_0 = \text{id}$ ,  $F_s F_t = F_{s+t}$  ensures you are the flow for the v.f.  $X|_p = \frac{\partial}{\partial t} |_{t=0} F_t(p)$  since:

(for manifold  $M$   
and smooth map  
 $F: \mathbb{R} \times M \rightarrow M$ .  
Call  $F_t = F(t, \cdot)$ )

$$\frac{\partial}{\partial t} |_t F_t(p) = \frac{\partial}{\partial s} |_0 F_{t+s}(p) \stackrel{\text{chain rule}}{=} \frac{\partial}{\partial s} |_0 F_s(F_t(p)) = X|_{F_t(p)} \quad \checkmark \quad \blacksquare$$

Def The Lie group homomorphisms  $\mathbb{R} \xrightarrow{\gamma} G$  are called 1-parameter subgroups of  $G$

Cor

$$\begin{array}{ccc} \text{Lie } G & \xrightarrow{\text{v.s.}} & T_1 G \\ X & \longleftrightarrow & v = X|_1 = \gamma'_v(0) \end{array} \xrightarrow{\text{bijection}} \{ \text{1-parameter subgroups of } G \}$$

Def Exponential map

$$\exp : \text{Lie } G \cong T_1 G \longrightarrow G$$

$$v \longmapsto \gamma_v(1)$$

Next time:

$\psi : \mathbb{R}^n \cong T_1 G \xrightarrow{\exp} G$  is a parametrization since  $\exp$  is smooth, and invertible near  $0 \in T_1 G$ .

- ① determined by conditions,  $\gamma_v(s+t) = \gamma_v(s)\gamma_v(t) \forall s, t$ ,  $\gamma_v'(0) = v$
- ② also determined by equation:  $\gamma_v(0) = 1$ ,  $\gamma_v'(t) = X|_{\gamma_v(t)}$ .

(flow for time 1 in direction  $X$  where  $X$  is the left-invt vector field corresponding to  $v \in T_1 G$  (so  $X|_g = D_g \psi \cdot v$ ))

## EXAMPLE 1

Torus  $G = \mathbb{R}^n / \mathbb{Z}^n$ : obvious 1-param. subgrps  $\gamma_v(t) = t v \bmod \mathbb{Z}^n$

Check condition ①:

$$\gamma_v(s+t) = (s+t)v = sv + tv \text{ hence } \gamma_v : \mathbb{R} \rightarrow G \text{ homomorphism.}$$

This classifies all 1-param. subgrps since  $\gamma_v'(0) = v \in \mathbb{R}^n \cong T_1 G$  is general.

$$\Rightarrow \exp(v) = v \bmod \mathbb{Z}^n$$

$\Rightarrow \exp : \mathbb{R}^n \rightarrow G$  is the map  $\Pi$  of questionsheet 1  
 $v \longmapsto v \bmod \mathbb{Z}^n$

Alternative approach: check condition ② for  $\gamma_v(t) = tv \bmod \mathbb{Z}^n$

$$\gamma_v(0) = 0, \quad \gamma'_v(0) = v, \quad \gamma'_v(t) = v$$

For the last two equalities we used the parametrization  $\Pi : \mathbb{R}^n \rightarrow G$   
 $\Pi(v) = v \bmod \mathbb{Z}^n$

Recall from Question sheet 1 that the left-invt v.f. on  $T^n$  in the parametr.  $\Pi$  are the constant vectors  $v \in \mathbb{R}^n$ .

Case n=1  $G = \mathbb{R}/\mathbb{Z} = S^1$  get  $\exp(x) = e^{2\pi i x} \in S^1$

Remark

here we identify  $T_1(\mathbb{R}/\mathbb{Z}) \cong \mathbb{R}$  via  $[ \begin{matrix} \text{curve} \\ t \mapsto tv \bmod \mathbb{Z} \end{matrix} ] \leftrightarrow v$

so we must get  $\exp(\mathbb{Z}) = 1$ . This corresponds to parametrizing the circle with " $x \in [0, 1]$ " rather than with " $\theta \in [0, 2\pi]$ ".

If you instead parametrize with  $\theta \mapsto e^{i\theta}$  then  $\exp(\theta) = e^{i\theta} \in S^1$ .

## EXAMPLE 2

$$G = GL(n, \mathbb{R})$$

$$\gamma_B(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} B^n$$

$$\exp(B) = \sum \frac{1}{n!} B^n$$

$$B \in \text{Mat}_{n \times n}(\mathbb{R}) \\ = T_1 GL(n, \mathbb{R})$$

CLAIM makes sense (converges) & can differentiate term by term

proof below

$$\Rightarrow \gamma_B(t) = I + tB + \frac{1}{2!} t^2 B^2 + \frac{1}{3!} t^3 B^3 + \dots$$

$$\gamma'_B(t) = B + tB^2 + \frac{1}{2!} t^2 B^3 + \dots$$

$$\Rightarrow \gamma_B(0) = I, \quad \gamma'_B(0) = B, \quad \gamma'_B(t) = \left. \gamma_B(t) \cdot B \right|_{\gamma_B(t)} = X \quad \text{for } X = \text{left-inv.}$$

$$\Rightarrow 1\text{-param subgp for } B \quad \blacksquare \quad \text{v.f. for } B: X|_x = x \cdot B$$

$$\text{compare } \frac{\partial}{\partial t} e^{tx} = e^{tx} \cdot x$$

finite dim'l, so all norms are equivalent

proof of claim The vector space  $\text{Mat}_{n \times n}(\mathbb{R})$  is a normed space using the norm

$$\|B\| = n \cdot \max_{i,j} |B_{ij}|$$

Nice properties:

- $\|BC\| \leq \|B\| \cdot \|C\|$
- complete, i.e. Cauchy sequences converge. (since finite dim'l)
- $\text{Mat}_{n \times n}(\mathbb{R})$  is an algebra

$$\begin{aligned} \|BC\| &= n \cdot \max_{i,k} |\sum_j B_{ij} C_{jk}| \\ &\leq n \cdot \max_{i,k} \underbrace{\sum_j |B_{ij}|}_{\leq \|B\|} \cdot \underbrace{|C_{jk}|}_{\leq \|C\|} \leq \frac{\|B\|}{n} \|C\| \\ &\leq \cancel{n} \cdot \cancel{n} \frac{\|B\|}{\cancel{n}} \frac{\|C\|}{\cancel{n}} \end{aligned}$$

(v.s.  $V$  with bilinear  $V \times V \rightarrow \mathbb{R}$ ) defining multiplication

Def Complete normed algebras satisfying  $\star$  are called Banach algebras

For Banach algebras can reprove all the usual results about series, absolute convergence, radius of convergence, etc.  $\square$  (SAME PROOFS!)

If  $I \in \text{Banach}$   $\Rightarrow \exp(tx) = I + tx + \frac{t^2}{2!} x^2 + \dots$  converges absolutely and radius of convergence  $= \infty$

$\Rightarrow$  can differentiate in  $t$  term by term  $\blacksquare$