

LECTURE 4

LAST TIME :

$$\begin{array}{c} \text{Lie } G \xrightleftharpoons[\text{v.s.}]{\cong} T_1 G \xrightleftharpoons[\text{bijection}]{\cong} \{\text{l-parametr-subgroups}\} \\ X \longleftrightarrow v \longleftrightarrow (\gamma_X = \gamma_v : \mathbb{R} \rightarrow G) \end{array}$$

recall : $\begin{cases} \gamma_v(0) = 1 \\ \gamma'_v(0) = v = X|_1, \\ \gamma'_v(t) = X|_{\gamma_v(t)} \end{cases}$ also recall that
 IF $\begin{cases} \gamma \text{ smooth} \\ \gamma : \mathbb{R} \rightarrow G \text{ hom} \\ (\gamma(s+t) = \gamma(s)\gamma(t)) \end{cases}$ THEN $\gamma = \gamma_v$ where $v = \gamma'(0)$.

THE EXPONENTIAL MAP

Def $\exp : \text{Lie } G \cong T_1 G \rightarrow G$, $\exp(v) = \gamma_v(1)$.

Lemma $\exp(sv) = \gamma_v(s)$

If $t \mapsto \gamma_{sv}(t)$, $t \mapsto \gamma_v(st)$ are homs $\mathbb{R} \rightarrow G$, so just compare $\frac{\partial}{\partial t}|_{t=0}$:

$\frac{\partial}{\partial t}|_0 \gamma_{sv}(t) = sv$, $\frac{\partial}{\partial t}|_0 \gamma_v(st) = s \cdot \gamma'_v(s \cdot 0) = sv$. So by uniqueness $\gamma_{sv}(t) = \gamma_v(st)$
 chainrule \Rightarrow claim, taking $t=1$ ■

Theorem $\exp : \text{Lie } G \rightarrow G$ is smooth

side remark: vector spaces V are manifolds: just pick a basis to get a (global!) parametrization $\mathbb{R}^n \cong V$. Their tangent spaces are:
 $T_v V \cong V$, $[curve \gamma(t) = v + tw] \mapsto w = \gamma'(0)$
 Hence Lie G is a manifold.

If \exp is the composite of 3 smooth maps:

$$\begin{array}{ccccccc} \text{Lie } G & \longrightarrow & \mathbb{R} \times (G \times \text{Lie } G) & \longrightarrow & G \times \text{Lie } G & \longrightarrow & G \\ X & \longmapsto & (1, 1, X) & \longmapsto & (t, g, X) & \longmapsto & (g, X) \longmapsto g \end{array}$$

flow of γ on $G \times \text{Lie } G$ where $\gamma|_{(g,X)} = (X|_g, 0)$ ■
 hence smooth

Lemma $D_0 \exp = \text{Id}$

$$\text{Pf } D_0(\exp) \cdot w = \frac{\partial}{\partial s}|_{s=0} \exp(0 + sw) = \frac{\partial}{\partial s}|_{s=0} \gamma_{sw}(1) \stackrel{\text{above Lemma}}{=} \frac{\partial}{\partial s}|_0 \gamma_w(s) = \gamma'_w(0) = w \blacksquare$$

Cor $\exp : \text{Lie } G \rightarrow G$ is a local diffeomorphism near 0.

$(\exists \text{ open sets } \overset{0}{U} \subseteq \text{Lie } G, \overset{0}{V} \subseteq G \text{ s.t. } \exp : U \rightarrow V \text{ is a diffeomorph.})$

Pf

FACT Inverse Function Theorem $\varphi : M \rightarrow N$ smooth map of mfds

$D_m \varphi : T_m M \rightarrow T_{\varphi(m)} N$ invertible $\Rightarrow \varphi$ local diffeo near m ■

Cor $\psi : \mathbb{R}^n \cong T_1 G \cong \text{Lie } G \xrightarrow{\exp} G$ is a parametrization near $1 \in G$
 choice of basis ($GL(n, \mathbb{R})$ choices)
 (defined on a nbhd of $0 \in \mathbb{R}^n$)
 (Hence get nice parametrizations $\phi_g \circ \psi \dots$)

EXAMPLES

1) $\exp : \mathbb{R} \rightarrow S^1$, $\exp(x) = e^{2\pi i x}$ local diffeo near 0, but not global (not injective!).
 a local inverse near $e^{2\pi i \cdot 0} = 1$ is $\frac{1}{2\pi i} \log(y) \leftarrow$ pick a branch of complex log.
 ($2\pi i \mathbb{Z}$ choices)

2) $\exp : \text{Mat}_{n \times n}(\mathbb{R}) \rightarrow \text{GL}(n, \mathbb{R})$, $\exp(B) = \sum \frac{1}{n!} B^n$, then a local inverse near I is
 $A \mapsto \log(A) = \log(I + (A - I))$ (for $\|A - I\| < 1$)

EXPLANATION (NON-EXAMINABLE):

For a Banach algebra with 1 define

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} x^n \text{ for } \|x\| < 1$$

radius of convergence

Remark we know $\exp(\log(1+tx)) = 1+tx$ for $t, x \in \mathbb{R}$ is an absolutely convergent series in tx (where defined). Hence same must be true for Banach algebras with 1. The reason is: take $x = 1 \in$ Banach algebra and $t \in \mathbb{R}$. Then the coefficients of those series in $t \cdot 1$ must agree with the coefficients of the series you got when working with \mathbb{R} ! (consider the coefficients inductively letting $t \rightarrow 0$ allows you to ignore higher order terms).

Def $\varphi : G \rightarrow H$ Lie group homomorphism means

- 1) φ group homomorphism
- 2) φ smooth

Theorem (Naturality of \exp)

If $\varphi : G \rightarrow H$ Lie group hom then:

$$\begin{array}{ccc} T_1 G & \xrightarrow{D_1 \varphi} & T_1 H \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{\varphi} & H \end{array} \quad \text{commutes}$$

(i.e. composing $\xrightarrow{\quad}$
 equals composing \downarrow)

Pf $v = [\gamma_v(t)] \longmapsto D_1 \varphi \cdot v = [\varphi \circ \gamma_v(t)]$
 $(\text{since } \gamma'_v(0) = v)$ $\gamma_v(1) \longmapsto \varphi \circ \gamma_v(1)$

Note $\varphi \circ \gamma_v(t)$ is a 1-parameter group since φ is a group hom. Hence \exp is evaluation at $t=1$. ■

EXAMPLES ① $\mathbb{R} \cong TS^1 \longrightarrow \mathbb{C} = \text{Mat}_{1 \times 1}(\mathbb{C})$

$$\mathbb{R}/\mathbb{Z} = S^1 \longrightarrow \mathbb{C}^* = \mathbb{C} \setminus \{0\} = \text{GL}(1, \mathbb{C})$$

(Note: $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$ is $\exp(z) = e^z$ since $= \sum \frac{1}{n!} z^n$ as 1×1 matrix)

$$x \longmapsto z = 2\pi i x$$

$$x \bmod \mathbb{Z} \longmapsto e^z = e^{2\pi i x}$$

② $\text{SkewSym}_{n \times n}(\mathbb{R}) \rightarrow \text{Mat}_{n \times n}(\mathbb{R})$

$$\downarrow \text{inclusion} \quad \downarrow$$

$$\text{O}(n) \xrightarrow{\text{inclusion}} \text{GL}(n, \mathbb{R})$$

implies that $\exp(B) = \sum \frac{1}{n!} B^n$ also for $\text{O}(n)$

(indeed for any Lie subgroup of $\text{GL}(n, \mathbb{R})$, not just $\text{O}(n)$)