

COVERING MAPS

A smooth surjective map of manifolds $\pi: M \rightarrow N$ is a covering map if there is an open set U around any point $p \in N$ with:

- $\pi^{-1}(U) = \bigsqcup \tilde{U}_i$ disjoint union of open sets (the sheets over U)
- $\tilde{U}_i \xrightarrow{\pi} U$

The fibre over p is $\pi^{-1}(p) = \bigsqcup \{\tilde{p}_i\}$ (discrete set)

EXAMPLES

- $\mathbb{R} \xrightarrow{e^{2\pi i x}} S^1$, pictorially:  pictorially: $\exp: \mathbb{R} \rightarrow S^1$, $\exp^{-1}(1) = \mathbb{Z}$
- Q. sheet 1: $\exp = \pi: \mathbb{R}^n \rightarrow T^n = \mathbb{R}^n / \mathbb{Z}^n$ cover, fibre $\cong \mathbb{Z}^n$
- Q. sheet 2: $SU(2) \rightarrow SO(3)$ double cover, fibre $\cong \{\pm I\}$
- NON-EXAMPLE: $(0, 3) \xrightarrow{e^{2\pi i x}} S^1$ local diffeo but not covering

Rmks

- π local diffeo
- If N is connected, the fibres are all homeomorphic
(pick a path γ from p to p' in $N \Rightarrow \pi^{-1}(\gamma)$ are paths from \tilde{p}_i to \tilde{p}'_i)
- FACT If M, N compact mfds of same dimension, N connected then
 $M \xrightarrow{\pi} N$ covering $\Leftrightarrow \pi$ local diffeo $\Leftrightarrow D\pi$ surjective

Lemma For $\pi: H \rightarrow G$ Lie group hom and covering, then

- | | | |
|------------------|------|--|
| $\cdot 1$ | V | i) $\text{Ker } \pi = \pi^{-1}(1)$ is a discrete closed normal subgp of H |
| $\cdot h$ | hV | ii) fibres are homeomorphic to $\text{Ker } \pi$ |
| $\downarrow \pi$ | | iii) for small enough nbhd V of $1 \in H$, |
| $\cdot 1$ | U | $\bigsqcup_{k \in \text{Ker } \pi} k \cdot V \rightarrow U = \pi(V)$ are the sheets over U |
| | | iv) $H/\text{Ker } \pi \cong G$ (by 1st iso theorem) |

Pf for (ii): $\pi^{-1}(g) = h \cdot \text{Ker } \pi$ if $\pi(h) = g$

for (iii): pick $U \ni 1$ as in definition of covering, $V = \tilde{U}$ sheet containing 1.
 \tilde{U}' another sheet $\Rightarrow \tilde{U}' \cap \text{Ker } \pi = \{k\}$ some $k \Rightarrow \tilde{U}' = kV$ ■

Theorem 1 $H \xrightarrow{\pi} G$ Lie gp hom, G connected, then

π covering $\Leftrightarrow D\pi: \mathfrak{h} \rightarrow \mathfrak{g}$ isomorphism

Pf of " \Rightarrow ": π covering $\Rightarrow \pi$ local diffeo near 1 $\Rightarrow D\pi: T_1 H \rightarrow T_1 G$ iso ■

Pf of " \Leftarrow ": $D\pi$ iso $\Rightarrow D_h \pi$ iso, all $h \in H$
(Lecture 5)

$\Rightarrow \pi$ local diffeo $\Rightarrow \pi^{-1}(1) = \text{Ker } \pi$ discrete ①

inverse function thm $\Rightarrow \text{image}(\pi)$ is subgrp of G containing nbhd of 1 (G connected) $\Rightarrow \pi$ surjective ②

Trick $H \times H \rightarrow H$
 $(h, l) \mapsto h^{-1}l$ smooth $\Rightarrow \exists$ nbhd V of $1 \in H$ with $(V^{-1} \cdot V) \cap \text{Ker } \pi = \{1\}$

Claim $\bigsqcup_{k \in \text{Ker } \pi} h \cdot k \cdot V \xrightarrow{\pi} g \cdot \pi(V) = \text{nbhd of } 1 \in G$ whenever $\pi(h) = g$
(use ②)
 $h \cdot k \cdot V \xrightarrow{\pi} g \cdot \pi(V)$ are the sheets over $g \cdot \pi(V)$.

Pf similar to pf of Lemma, in particular:

$h \cdot k \cdot V \xrightarrow{\pi} g \cdot \pi(V)$ local diffeo ✓

surjective ✓

injective since: $\pi(v_1) = \pi(v_2) \Rightarrow \pi(\underbrace{v_1^{-1} v_2}_{} \cdot 1) = 1$ so $v_1^{-1} v_2 = 1 \in V^{-1} \cdot V$

hence diffeo \blacksquare

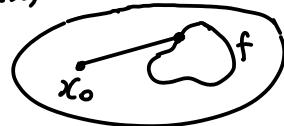
SIMPLY-CONNECTED GROUPS

A path-connected manifold is simply-connected if continuous maps $S' \xrightarrow{f} M$ are contractible
 \hookrightarrow connected mfd
see Lecture 2 meaning: \exists continuous $F: S' \times [0,1] \rightarrow M$
 $F(\cdot, 0) = f$, $F(\cdot, 1) = \text{constant}$

EXAMPLES

- \mathbb{R}^n : $F(x, t) = (1-t)f(x)$

- Convex subsets of \mathbb{R}^n



$$F(x, t) = t x_0 + (1-t)f$$

- $\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \xrightarrow{\text{diffeo}} \mathbb{R}^3$

- $S^n \subseteq \mathbb{R}^{n+1}$ spheres ($n \geq 2$) \leftarrow FACT it's simply connected

- $SU(2) \cong S^3$

- $SU(n)$, $SL(n, \mathbb{C})$ \leftarrow FACT simply connected.

e.g. $GL(1, \mathbb{C}) = \mathbb{C} \setminus \{0\}$

NON-EXAMPLES $S^1, T^n, SO(n), U(n), SL(n, \mathbb{R})$ ($n \geq 2$), $GL(n, \mathbb{C})$

A covering $M \rightarrow N$ is called universal cover if M is simply connected.
 \nwarrow (path connected)

FACT Universal covers exist and are unique up to diffeomorphism.

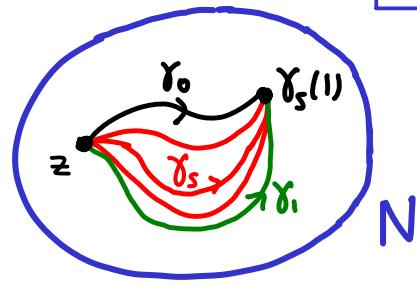
Sketch of existence (Non-examinable)

Fix $z \in N$ $M = \{ [\gamma] : \gamma: [0,1] \rightarrow N \text{ continuous path}, \gamma(0) = z \}$

equivalence class: identify paths γ_0, γ_1 if can continuously deform γ_0 to γ_1 keeping endpoints fixed

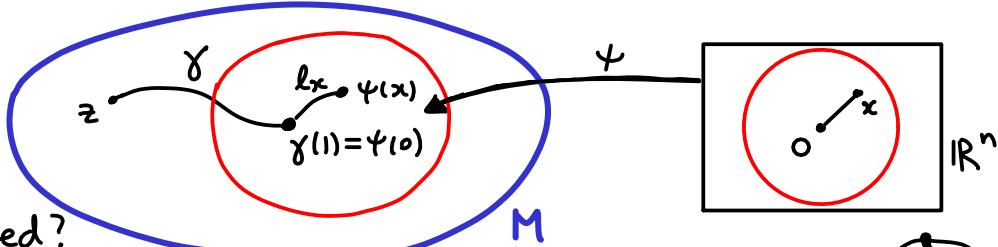
$(F: [0,1] \times [0,1] \xrightarrow{\text{cts}} N, F(\cdot, 0) = \gamma_0, F(\cdot, 1) = \gamma_1, F(0, \cdot) = z, F(1, \cdot) = \gamma_0(1) = \gamma_1(1))$

(think of this as a continuous family of paths $\gamma_s = F(\cdot, s)$)



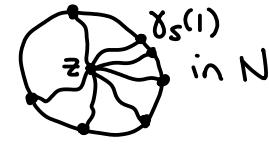
$$M \xrightarrow{\pi} N, \pi([\gamma]) = \gamma(1) = \text{end point of } \gamma$$

Rmk A local parametrization $\mathbb{R}^n \xrightarrow{\psi} N$ near $\psi(0) = \gamma(1)$ also parametrizes M near $[\gamma]$ via $[\gamma \# l_x]$ attach straight line segment $\gamma(tx)_{0 \leq t \leq 1}$ to γ so $\pi[\gamma \# l_x] = \psi(x)$.



Why M simply-connected?

Idea: a loop $S^1 \rightarrow M, s \mapsto [\gamma_s]$ corresponds to a picture of form so just contract it down to \bar{z} by $F(s, r) = [\text{path } t \mapsto \gamma_s((1-r) \cdot t)]$.



Lemma For G Lie group, the universal cover \tilde{G} is a Lie group in a natural way so that $\tilde{G} \xrightarrow{\pi} G$ is surj. Lie group hom, hence $\text{Lie}(\tilde{G}) \xrightarrow{D_1 \pi} \mathfrak{g}$ Lie algebra iso

Proof In the construction of \tilde{G} above pick $\underline{z=1}$. also $\tilde{G}/\ker \pi \cong G$.

unit: $\tilde{1} = [\text{constant path at 1}]$

multiplication: $[\gamma_1] \cdot [\gamma_2] = [\text{path } t \mapsto \gamma_1(t) \cdot \gamma_2(t)]$ notice via π these are the operations in N : $\gamma_1(1) \cdot \gamma_2(1)$ and $\gamma(1)^{-1}$
inversion: $[\gamma]^{-1} = [\text{path } t \mapsto \gamma(t)^{-1}]$ \Rightarrow locally, in above parametrizations, the operations are smooth since smooth in N .

EXAMPLES OF $\tilde{G} \rightarrow G$

- $\mathbb{R} \xrightarrow{\exp} S^1$ and $\mathbb{R}^n \xrightarrow{\exp} T^n$
- $SU(2) \rightarrow SO(3)$ Q.sheet 2
- $\text{Spin}(n) = \widetilde{SO(n)}$ definition of spin group for $n > 3$ (example: $\text{Spin}(3) \cong SU(2)$)
FACT $\text{Spin}(n) \rightarrow SO(n)$ is a double cover

FACT "Can't make universal covers any larger":

$\pi: \tilde{M} \rightarrow M$ covering, M simply conn. $\Rightarrow \pi$ diffeo
 \tilde{M} connected

Non-examinable proof idea: suffices to show fibre $\pi^{-1}(m)$ is a point.

By contradiction, if $\tilde{m}_1, \tilde{m}_2 \in \pi^{-1}(m)$, a curve connecting \tilde{m}_1, \tilde{m}_2 would give a contractible loop in M via π . Lifting this contraction to \tilde{M} then shows $\tilde{m}_1 = \tilde{m}_2$.

Cor $\tilde{G} \xrightarrow{\pi} G$ Lie gp hom + covering $\Rightarrow \pi$ Lie group iso
connected \uparrow simply connected (and connected) using that fibres are discrete

CORRESPONDENCE BETWEEN LIE ALG. & LIE GP. HOMOMORPHISMS

Theorem If H is simply-connected (and connected),

$$\left\{ \begin{array}{l} \text{Lie algebra homs} \\ g \xrightarrow{\psi} \mathfrak{g} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{Lie group homs} \\ H \xrightarrow{\varphi} G \end{array} \right\}$$

$$\psi = D_1 \varphi \quad \varphi$$

Proof Recall Lecture 5 : a Lie gp hom $H \xrightarrow{\varphi} G$ is uniquely determined by D, φ .
 Remains to show existence.

$$\psi: \mathfrak{h} \rightarrow \mathfrak{g} \Rightarrow \text{graph } \Gamma = \{(x, \psi(x)) : x \in \mathfrak{h}\} \subseteq \mathfrak{h} \times \mathfrak{g}$$

is a Lie subalgebra since ψ Lie alg hom.

(Chevalley) \Rightarrow corresponds to a connected Lie subgroup $S \subseteq H \times G$, $\text{Lie}(S) = \Gamma$.

Consider the projection $H \times G \rightarrow H$ Lie gp hom $\Rightarrow \mathfrak{h} \oplus \mathfrak{g} \rightarrow \mathfrak{h}$ Lie alg hom

Observe : $\pi: S \subseteq H \times G \rightarrow H$ has $D, \pi: \Gamma \subseteq \mathfrak{h} \oplus \mathfrak{g} \rightarrow \mathfrak{h}$ isomorphism

(Theorem 1)

$\Rightarrow \pi: S \rightarrow H$ covering

(Corollary) " diffeo (using H simply connected)

$\Rightarrow H \xrightarrow{\pi^{-1}} S \subseteq H \times G \xrightarrow{\text{project}} G$ Lie gp hom inducing ψ since :

$$\mathfrak{h} \longrightarrow \Gamma \subseteq \mathfrak{h} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$x \longmapsto (x, \psi(x)) \longrightarrow \psi(x) \quad \blacksquare$$

Cor H, G simply-connected Lie gps with $\mathfrak{h} \cong \mathfrak{g} \Rightarrow H \cong G$ iso Lie gps

Pf $\mathfrak{h} \xrightarrow{\varphi} \mathfrak{g}$ gives a unique $H \xrightarrow{\varphi} G$

$\mathfrak{g} \xrightarrow{\varphi'} \mathfrak{h}$ " " $G \xrightarrow{\varphi'} H$

$\mathfrak{h} \xrightarrow{\varphi} \mathfrak{g} \xrightarrow{\varphi'} \mathfrak{h}$ " " $H \longrightarrow H$ which must be both $\varphi' \circ \varphi$ and identity $\} \Rightarrow \varphi' \circ \varphi = \text{id}$ $\} \Rightarrow \varphi$ iso \blacksquare

FACT Ado's theorem For any Lie algebra V , there is an injective Lie algebra hom $V \longrightarrow \text{gl}(m, \mathbb{R})$, some m .

Lie's third theorem

There is a 1-to-1 correspondence

$$\left\{ \begin{array}{l} \text{Lie algebras} \\ \text{isos} \end{array} \right\} / \text{Lie alg isos} \quad \xleftrightarrow{1:1} \quad \left\{ \begin{array}{l} \text{simply-connected} \\ \text{Lie groups} \end{array} \right\} / \text{Lie gp isos}$$

Pf By Cor, get uniqueness of G (up to iso) for given \mathfrak{g} (up to iso)

Remains to show existence of G given Lie algebra V .

Ado's thm $\Rightarrow V \subseteq \text{gl}(m, \mathbb{R})$ Lie subalg

(Chevalley) \exists connected Lie subgp $H \subseteq \text{GL}(m, \mathbb{R})$ with $\mathfrak{h} = V$

\Rightarrow take $G = \widetilde{H}$ universal cover : simply connected and $\mathfrak{g} \cong \mathfrak{h} = V$ \blacksquare