

REPRESENTATION THEORYLecture 5: Representation  $V$  of Lie group  $G$  means:Continuous hom  $G \rightarrow \text{Aut}(V)$  where  $V$  vector space

- Rmk 1)
- we work over field  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$
  - always assume  $V$  finite-dimensional v.s./ $\mathbb{F}$ ,  $d = \dim_{\mathbb{F}}(V)$
- 2)  $\text{Aut}(V) = \{\text{linear bijections } V \rightarrow V\}$ . If pick basis of  $V$ ,  $\text{Aut}(V) \cong \text{GL}(d, \mathbb{F})$
- 3) Recall continuous hom  $\Rightarrow$  smooth hom

Lemma Equivalent definition of rep:Group action  $G \times V \rightarrow V$  continuous in  $G$ , linear in  $V$ 

Explicitly:

i)	$1 \cdot v = v$	all $v \in V$
ii)	$(g_1 g_2) \cdot v = g_1 \cdot (g_2 \cdot v)$	all $g_1, g_2 \in G, v \in V$

For this reason, we often call  $V$  a  $G$ -module or  $G$ -mod

Pf Action determines  $G \xrightarrow{\varphi} \text{Bijections}(V, V)$ ,  $g \mapsto (v \mapsto g \cdot v)$   
 continuous in  $G$  and linear in  $V$ , hence  $G \xrightarrow{\varphi} \text{Aut}(V)$  rep.

Conversely, for rep  $G \xrightarrow{\varphi} \text{Aut}(V)$  define action  $g \cdot v = \varphi(g)(v)$  ■Lecture 5: Representation  $V$  of Lie algebra  $\mathfrak{g}$  means:Lie algebra hom  $\mathfrak{g} \rightarrow \text{End}(V)$  where  $V$  vector spaceusing the bracket  $[\alpha, \beta] = \alpha \circ \beta - \beta \circ \alpha$  on  $\text{End}(V)$ Rmk  $\text{End}(V) = \{\text{linear maps } V \rightarrow V\}$ . If pick basis of  $V$ ,  $\text{End}(V) \cong \text{Mat}_{d \times d}(\mathbb{F})$ Lecture 10  $\Rightarrow$  For  $G$  simply-connected (and connected) Lie gp:

$$\{\text{Lie gp reps } G \rightarrow \text{Aut}(V)\} \xleftrightarrow{1:1} \{\text{Lie alg reps } \mathfrak{g} \rightarrow \text{End}(V)\}$$

Q.5 Question sheet 4 (case  $SU(2) \rightarrow SO(3)$ ) generalizes to universal covers  $\widetilde{G} \xrightarrow{\pi} G$ :  
 For  $G$  connected Lie gp:

$$\{\text{Lie gp reps } \widetilde{G} \xrightarrow{\tilde{\varphi}} \text{Aut}(V)\} \xleftrightarrow{1:1} \{\text{Lie alg reps } \mathfrak{g} \xrightarrow{\psi} \text{End}(V)\}$$

For rep  $G \xrightarrow{\varphi} \text{Aut}(V) \Rightarrow$  get rep  $\tilde{\varphi} = \varphi \circ \pi: \widetilde{G} \xrightarrow{\pi} G \xrightarrow{\varphi} \text{Aut}(V)$  conversely given  
 $\Rightarrow$  forces  $\ker \pi \subseteq \ker \tilde{\varphi}$  (i.e.  $\ker \pi \subseteq \widetilde{G}$  acts by Id on  $V$ )  $\tilde{\varphi}: \widetilde{G} \rightarrow \text{Aut}(V)$  with  $\ker \pi \subseteq \ker \tilde{\varphi}$  then get  
 $\psi: \mathfrak{g} \cong \widetilde{G}/\ker \pi \xrightarrow{\tilde{\varphi}} \text{Aut}(V)$

$$\Rightarrow \{\text{Lie gp reps } G \rightarrow \text{Aut}(V)\} \xleftrightarrow{1:1} \{\text{Lie gp reps } \widetilde{G} \xrightarrow{\tilde{\varphi}} \text{Aut}(V) \text{ with } \ker \pi \subseteq \ker \tilde{\varphi}\}$$

## EXAMPLES

- Trivial representation  $G \rightarrow \text{Aut}(V)$ ,  $g \mapsto \text{Id}$
- Adjoint representation  $\text{Ad}: G \rightarrow \text{Aut}(g)$

Def faithful rep means injective rep.

In practice it means you can identify  $G$  or  $g$  with a subset of matrices.

## EXAMPLES Standard representations

$$\left. \begin{array}{l} O(n) \rightarrow \text{Aut}(\mathbb{R}^n) \\ U(n) \rightarrow \text{Aut}(\mathbb{C}^n) \\ \dots \end{array} \right\} \text{given by left-multiplication: } A \longmapsto \left( \begin{array}{c} \mathbb{F}^n & \xrightarrow{\quad} & \mathbb{F}^n \\ v & \longmapsto & A \cdot v \end{array} \right)$$

Def  $V, W$   $G$ -mods, a  $G$ -linear map (or  $G$ -mod homomorphism) means

- $\mathbb{F}$ -linear map  $f: V \rightarrow W$
- $f$  commutes with  $G$ -action:  $f(g \cdot v) = g \cdot f(v)$

$$\text{Hom}_G(V, W) = \{G\text{-linear maps } f: V \rightarrow W\}$$

Def  $V, W$  equivalent reps if  $\exists$   $G$ -isomorphism  $f: V \rightarrow W$

Write  $\underline{V \cong W}$ .

$\nwarrow$  (bijective  $G$ -linear map)

Explicitly:

$$\rho_i: G \rightarrow \text{Aut}(\mathbb{R}^n) = GL(n, \mathbb{R}) \text{ reps}$$

$f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  iso  $\Rightarrow$  given by invertible matrix  $F$

$$\rho_i \text{ equivalent} \Leftrightarrow f(\rho_i(g)v) = \rho_2(g)f(v) \Leftrightarrow \rho_i(g) = F^{-1} \cdot \rho_2(g) \cdot F$$

$\Leftrightarrow \rho_i(g)$  are conjugate via an iso  $F$ , for all  $g$

## INVARIANT INNER-PRODUCTS

Def For a rep  $V$ , an inner product (Hermitian i.p. if  $\mathbb{F} = \mathbb{C}$ )  $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{F}$  is  $G$ -invariant if

$$\langle g v, g w \rangle = \langle v, w \rangle \quad \text{all } g \in G, v, w \in V$$

and call  $V$  an orthogonal rep ( $\mathbb{F} = \mathbb{R}$ ) or unitary rep ( $\mathbb{F} = \mathbb{C}$ ).

Explicitly If pick o.n./unitary basis  $e_i$  for  $V$

$$\Rightarrow \langle \sum a_i e_i, \sum b_j e_j \rangle = a^* b \quad (a, b \in \mathbb{F}^n \text{ and } * = \text{conjugate transpose})$$

$$\Rightarrow v^* \rho(g)^* \rho(g) w = \langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle = v^* w \quad \text{for all } v, w$$

$$\Rightarrow \rho(g)^* \rho(g) = \text{Id} \quad \text{so } \rho(g) \text{ is an orthogonal/unitary matrix}$$

$$\Rightarrow \mathbb{F} = \mathbb{R}: \rho: G \rightarrow O(n) \subseteq GL(n, \mathbb{R}) \cong \text{Aut}(V)$$

$$\mathbb{F} = \mathbb{C}: \rho: G \rightarrow U(n) \subseteq GL(n, \mathbb{C}) \cong \text{Aut}(V)$$

# NEW FROM OLD

Let  $V, W$  be  $G$ -mods : we want to build new  $G$ -mods from  $V, W$

- 1) direct sum  $V \oplus W$   $g(v, w) = (gv, gw)$
- 2) tensor product  $V \otimes_{\mathbb{F}} W$   $g(v \otimes w) = gv \otimes gw$

Recall :  $V \otimes W$  is a vector space of dimension  $\dim V \cdot \dim W$ .

We use the symbols  $v_i \otimes w_j$  to denote a basis of  $V \otimes W$ , whenever  $\{v_i\}$  is a basis of  $V$  and  $\{w_j\}$  is a basis of  $W$ .

EXAMPLE  $\mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{n \cdot m}$

It is convenient to extend the symbol  $\otimes$  to any vectors by declaring that:  $\leftarrow$  So  $v \otimes 0 = 0$  and  $0 \otimes w = 0$

$$(\sum \lambda_i v_i) \otimes (\sum \mu_j w_j) = \sum \lambda_i \mu_j (v_i \otimes w_j) \quad (\text{often call } v \otimes w \text{ generators})$$

Warning: not all vectors in  $V \otimes W$  arise as  $v \otimes w$  : you need to allow sums  $\sum_i v_i \otimes w_i$

For example in  $\mathbb{R}^2 \otimes \mathbb{R}^2$ ,  $e_1 \otimes e_2 + e_2 \otimes e_1 \neq v \otimes w$  for any  $v, w \in \mathbb{R}^2$ .

A linear map  $\varphi: V \otimes W \rightarrow U$  is determined by its values on generators,  $\varphi(v \otimes w)$ , since that determines  $\varphi$  on the basis  $v_i \otimes w_j$ . Conversely, if you define  $\varphi(v \otimes w)$  in a way that is linear in  $v$  and in  $w$ , then  $\varphi$  extends to a well-defined linear map  $V \otimes W \rightarrow U$ .

- 3)  $f \in \text{Hom}_G(V, W) \Rightarrow$  get  $G$ -mods :  $\text{Ker } f$   $f(v) = 0 \Rightarrow f(g \cdot v) = g \cdot f(v) = 0$   
 $\text{Im } f$   $g \cdot f(v) = f(g \cdot v) \in \text{Im } f$   
 $\text{Coker } f = W / \text{Im } f$   $g \cdot (w + \text{Im } f) = gw + \text{Im } f$

- 4) conjugate space  $\bar{V}$  : as sets  $\bar{V} = V$ , use same  $G$ -action, but change the  $\mathbb{C}$ -action:  
 $(\text{for } \mathbb{F} = \mathbb{C})$   $\lambda \cdot v = \bar{\lambda} v$  for  $\lambda \in \mathbb{C}$

- 5) dual space  $V^* = \{\text{IF-linear } V \xrightarrow{\psi} \mathbb{F}\}$   $(g \cdot \psi)(v) = \psi(g^{-1} \cdot v)$

Rmk : Need inverse  $g^{-1}$  to make it a left-action (check axiom (iii))

Another reason is that you want the following diagram to commute :

$$\begin{array}{ccc} V & \xrightarrow{\psi} & \mathbb{F} \\ g \downarrow & & \downarrow \text{id} \\ V & \xrightarrow{g \cdot \psi} & \mathbb{F} \end{array}$$

- 6) hom space  $\text{Hom}_{\mathbb{F}}(V, W) = \{\text{IF-linear } V \xrightarrow{\psi} W\}$   $(g \cdot \psi)(v) = g \cdot (\psi(g^{-1} \cdot v))$

This ensures that the following diagram commutes :

$$\begin{array}{ccc} V & \xrightarrow{\psi} & W \\ g \downarrow & & \downarrow g \\ V & \xrightarrow{g \cdot \psi} & W \end{array}$$

Lemma  $V^* \otimes_{\mathbb{F}} W \simeq \text{Hom}_{\mathbb{F}}(V, W)$   $G$ -iso

Pf  $\psi \otimes w \mapsto \left( \begin{array}{c} \psi: V \rightarrow W \\ \psi(v) = \underbrace{\psi(v)w}_{\in \mathbb{F}} \end{array} \right)$  and extend bilinearly in  $\psi$  and  $w$ .

- Linear by construction. ✓
- Bijective : pick basis  $v_i$  of  $V$ , get dual basis  $v_i^*$  so  $v_i^*(v_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$ . Pick basis  $w_i$  of  $W$ . Then  $v_i^* \otimes w_j$  is a basis for  $V^* \otimes W$ . The corresponding  $\psi$  maps are :  $\varphi_{ij}: V \rightarrow W$  with  $\varphi_{ij}(v_k) = v_i^*(v_k)w_j = \begin{cases} w_j & \text{if } i=k \\ 0 & \text{if } i \neq k \end{cases}$  so  $\varphi_{ij}$  is the matrix with  $\begin{cases} 1 & \text{in position } (j, i) \\ 0 & \text{in all other positions} \end{cases}$ . These matrices are a basis for  $\text{Hom}_{\mathbb{F}}(V, W)$ . ✓

- preserves  $G$ -action:

$$(g \cdot (\psi \otimes w))v = ((g\psi) \otimes (gw))v = (g\psi)(v) \cdot gw = \underbrace{\psi(g^{-1}v)}_{\in \mathbb{F}} gw = g(\psi(g^{-1}v)w) = (g \cdot \psi)(v). \quad \blacksquare$$

# REDUCIBILITY

Def A G-submodule or subrepresentation of  $V$  is a  $G$ -invariant vector subspace  $W \subseteq V$ .  
 Call  $V$  reducible if  $\exists$  subrep  $W \neq 0, V$        $\nwarrow G \cdot W \subseteq W$  (meaning:  $g \cdot w \in W$  for all  $g \in G, w \in W$ )  
irreducible if  $W = 0, V$  are the only subreps.  $\leftarrow$  call  $V$  irrep

## EXAMPLES

- $G = S^1$  acts on  $\mathbb{C}^2$  by  $e^{i\theta}: (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$ . This is reducible:  $\mathbb{C} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \subseteq \mathbb{C}^2$  is invariant.
- $G = \{(a b) : a, b \in \mathbb{R}, a \cdot c \neq 0\}$  acts naturally on  $\mathbb{R}^2$  ( $v \mapsto Av$  for  $v \in \mathbb{C}^2, A \in G$ ).  
 But  $\mathbb{R}^2$  is reducible:  $W = \mathbb{R} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is a  $G$ -submod since  $G \cdot W \subseteq W$ .
- $G = U(2)$  acting on  $\mathbb{C}^2$  is irreducible: if  $v \neq 0 \in \mathbb{C}^2$ , extend  $u_i = \frac{v}{|v|}$  to a unitary basis  $u_1, u_2$ . Then  $A = (u_1, u_2) \in U(2)$ , and  $A \cdot e_i = u_i = \frac{v}{|v|}$ . So  $A^{-1} \cdot v = |v|e_i$ . Similarly for  $B = (u_2, u_1)$ ,  $B^{-1} \cdot v = |v|e_2$ . So if  $W \subseteq \mathbb{C}^2$  is a subrep and  $v \neq 0 \in W$ , then  $G \cdot W \subseteq W$ , in particular  $A^{-1}v, B^{-1}v \in W$ , so  $e_1, e_2 \in W$ , so  $W = \mathbb{C}^2$  (since it is a vector subspace,  $\text{span}(e_1, e_2) \subseteq W$ )

Schur's Lemma  $f \in \text{Hom}_G(V, W)$  for  $V, W$  irreps  $\Rightarrow f$  iso or  $f = 0$

Pf  $\text{Ker } f = 0$  or  $V$  since  $V$  irrep,  $\text{Im } f = W$  or  $0$  since  $W$  irrep.  $\blacksquare$

Schur's Lemma over  $\mathbb{C}$   $f \in \text{Hom}_{\mathbb{C}}(V, V)$  for  $V$  irrep  $\Rightarrow f = \lambda \cdot \text{Id}$  some  $\lambda \in \mathbb{C}$

Pf  $\text{IF } \mathbb{F} = \mathbb{C} \Rightarrow \exists \lambda$  eigenvalue of  $f$   
 $\Rightarrow f - \lambda \cdot \text{Id} \in \text{Hom}_{\mathbb{C}}(V, W)$  with non-zero kernel  
 $\xrightarrow{\text{Schur}} f - \lambda \cdot \text{Id} = 0 \quad \blacksquare$

Cor  $V, W$  irreps  $\Rightarrow \begin{cases} \text{Hom}_G(V, W) = 0 & \text{if } V, W \text{ not equivalent} \\ \dim \text{Hom}_G(V, W) \geq 1 & \text{if } V, W \text{ equivalent} \\ & (= 1 \text{ when } \mathbb{F} = \mathbb{C}) \end{cases}$

Abbreviate  $nV = \underbrace{V \oplus \dots \oplus V}_{n \text{ copies}}$        $n \in \mathbb{N}$

Theorem  $V_i$  non-equivalent irreps, then  
 $\bigoplus m_i V_i \simeq \bigoplus n_i V_i \iff m_i = n_i$   
 (we assume only finitely many  $m_i, n_i$  are non-zero)

$\leftarrow$  IDEA: compare prime factorizations over  $\mathbb{N}$ .

$$\text{Pf} \Rightarrow: \text{Hom}_G(V_k, \bigoplus m_i V_i) \simeq \text{Hom}_G(V_k, \bigoplus n_i V_i)$$

$$\bigoplus m_i \text{Hom}_G(V_k, V_i) \stackrel{12}{\simeq} \bigoplus n_i \text{Hom}_G(V_k, V_i)$$

Schur:  $\text{Hom}_G(V_j, V_i) = 0$  unless  $i=j$

$$m_k \text{Hom}_G(V_k, V_k) \stackrel{12}{\simeq} n_k \text{Hom}_G(V_k, V_k)$$

Take dimensions  $\Rightarrow m_k = n_k \quad \blacksquare$