

This lecture:  $G$  COMPACT LIE GROUP,  $\mathbb{F} = \mathbb{C}$

REPRESENTATION RING = CHARACTER RING

Def

Representation ring  $R(G) = \left\{ \sum n_i V_i : n_i \in \mathbb{Z}, \text{finitely many } n_i \neq 0 \right\}$

using  $+$  and  $\otimes$

$V_i$ : non-equivalent irreps of  $G$ .

$(V_i \otimes V_j = \sum m_k V_k \text{ in } R(G) \text{ if } V_i \otimes_{\mathbb{C}} V_j \simeq \bigoplus m_k V_k)$  virtual reps or virtual  $G$ -mods

"honest" reps:  $\left\{ \sum n_i V_i \in R(G) : n_i \geq 0 \right\} \xleftrightarrow{1:1} \left\{ \text{equivalence classes of reps} \right\}$

(complete reducibility + last thm of Lecture 11)

Def  $C\ell(G) = \text{ring of class functions}$  : continuous  $G \xrightarrow{\text{f}} \mathbb{C}$  satisfying  $f(h^{-1}gh) = f(g)$

↑ pointwise addition & multiplication ↗ EXAMPLE characters  $f = \chi_V$ .

Thm  $\chi: R(G) \rightarrow C\ell(G)$ ,  $\chi(\sum n_i V_i) = \sum n_i \chi_{V_i}$  is an injective hom of rings

Pf Injective by orthogonality relns, hom since  $\chi_{V_i} \otimes_{\mathbb{C}} \chi_{V_j} = \chi_{V_i} \cdot \chi_{V_j}$  ◻

Def Often identify  $R(G)$  with  $\chi(R(G)) \leftarrow$  called character ring

Thm 1 Any class function is uniformly approximated by  $\sum z_i \chi_{V_i}$  ( $z_i \in \mathbb{C}$ )

(Q.Sheet 6) That is:  $\text{span}_{\mathbb{C}}(\text{Image } \chi) \subseteq C\ell(G)$  dense.

Rmk fails for  $\mathbb{F} = \mathbb{R}$ : holds if use  $\{f: G \rightarrow \mathbb{R} \text{ in } C\ell(G) \text{ with } f(g) = f(g^{-1})\}$

EXAMPLE  $G = S^1 = \mathbb{R}/\mathbb{Z}$   $\rho_1: S^1 \rightarrow GL(1, \mathbb{C})$   $\rho_a = \rho_1^{\otimes a}: S^1 \rightarrow GL(1, \mathbb{C})$

(here  $a \in \mathbb{Z}$ ,  $\rho_a = \rho_1^{\otimes |a|}$  if  $a < 0 \in \mathbb{Z}$ )  $\rho_1(x) = e^{2\pi i x} \cdot \text{Id}$   $\rho_a(x) = e^{2\pi i ax} \cdot \text{Id}$   
 $\chi_1(x) = e^{2\pi i x}$   $\chi_a(x) = e^{2\pi i ax}$

Q.sheet 5  $\Rightarrow R(S^1) = \left\{ \sum_{a \in \mathbb{Z}} n_a \rho_a \text{ finite sum, } n_a \in \mathbb{Z} \right\} \cong \mathbb{Z}[t, t^{-1}] = \text{Laurent polys in } t$

$\chi: R(S^1) \rightarrow C\ell(S^1)$ ,  $\chi(\sum n_a \rho_a) = \sum n_a e^{2\pi i ax} = \text{trigonometric polys with integer coeffs.}$

EXAMPLE  $G = T^n = \mathbb{R}^n / \mathbb{Z}^n$  Q.sheet 6 using  $T^n = S^1 x_1 \times \dots \times S^1$  (or directly by Q.sheet 5):

$R(G) = \left\{ \sum_{a \in \mathbb{Z}^n} n_a \rho_a \text{ finite sum, } n_a \in \mathbb{Z} \right\} \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] = \text{Laurent polys in } t_1, \dots, t_n$

$\rho_a: T^n \rightarrow GL(1, \mathbb{C})$ ,  $\rho_a(x) = e^{2\pi i \langle a, x \rangle} \cdot \text{Id}$ ,  $\chi_a(x) = e^{2\pi i \langle a, x \rangle}$   
where  $\langle a, x \rangle = a_1 x_1 + \dots + a_n x_n$

PETER-WEYL THEOREM

$\rho: G \rightarrow \text{Aut}(\mathbb{C}^n)$  gives a subset of  $C(G) = \{ \text{continuous functions } f: G \rightarrow \mathbb{C} \}$

called matrix entries:  $f(g) = ((i,j)\text{-entry of the matrix } \rho(g)) = \text{Trace } (\varphi \circ \rho(g))$

where  $\varphi: \mathbb{C}^n \rightarrow \mathbb{C}^n$  linear map with  $\varphi(e_i) = e_j$  and  $\varphi(e_k) = 0$  all  $k \neq j$ .  
 $(\varphi = \text{matrix with } 1 \text{ in position } (j,i) \text{ and } 0 \text{ elsewhere})$

EXAMPLE  $G = S^1 \subset \mathbb{C}^2$ ,  $\rho(x) = \begin{pmatrix} \cos 2\pi x & -\sin 2\pi x \\ \sin 2\pi x & \cos 2\pi x \end{pmatrix} \in \text{Aut}(\mathbb{C}^2)$ . Take  $(i,j) = (1,2)$   
 $\varphi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow \varphi \circ \rho(x) = \begin{pmatrix} 0 & 0 \\ \cos 2\pi x & -\sin 2\pi x \end{pmatrix} \Rightarrow \text{Trace}(\varphi \circ \rho) = -\sin 2\pi x = \begin{pmatrix} (1,2) \text{ entry} \\ \text{of } \rho(x) \end{pmatrix}$

FACT (Schur) matrix entries of irreps are orthogonal using  $\langle f_1, f_2 \rangle = \sum_{g \in G} \overline{f_1(g)} f_2(g)$ .

Rmk holds for  $\text{IF} = \mathbb{R}$  for two non-equivalent irreps, otherwise fails (e.g. example above).

EXAMPLE  $G = S^1$   $\dim(\text{irreps}) = 1 \Rightarrow$  matrix entry  $= \chi_a = e^{2\pi i ax}$  for  $a \in \mathbb{Z}$  are orthog.  
 (indeed orthogonality relns:  $\langle \chi_a, \chi_b \rangle = 0$  for  $a \neq b$ ). Above example fails since reducible/ $\mathbb{C}$ !

Def Representative function means any linear combination of matrix entries

They can always be written as  $L \circ \rho$  where  $\rho: G \rightarrow \text{Aut } V$ ,  $L \in \text{Hom}_{\mathbb{C}}(V, V)^*$

Let  $F(G) = \{\text{representative fns}\}$       "      "       $\text{Tr}(\varphi \circ \rho)$  for  $\rho: G \rightarrow \text{Aut } V$ ,  $\varphi \in \text{Hom}_{\mathbb{C}}(V, V)$

①  $L_1 \circ \rho_1 + \lambda L_2 \circ \rho_2 = (L_1 + \lambda L_2) \circ (\rho_1 + \rho_2)$  using  $V = V_1 \oplus V_2 \Rightarrow F(G)$  is v.s. / $\mathbb{C}$

② Product of two matrix entries from  $\rho_1, \rho_2$  is a matrix entry of  $\rho_1 \otimes \rho_2$  (Q.1 Q.Sheet 5)  
 $\Rightarrow F(G) \subseteq C(G)$  is subring and v.s./ $\mathbb{C}$ , so it's  $\mathbb{C}$ -algebra.

③ If only allow rep  $V$ , get vector subspace  $F_V(G) \subseteq F(G)$  of  $\dim \leq (\dim V)^2 = \#(\text{matrix entries})$

Above Fact  $\Rightarrow F(G) = \bigoplus F_{V_i}(G)$  orthogonal direct sum over the irreps  $V_i$

Peter-Weyl Theorem (version 1)  $F(G) \subseteq C(G)$  is dense (will not prove it.  
Really mostly functional analysis)

Rmk also holds for  $\text{IF} = \mathbb{R}$ .

$G \xrightarrow{f} \mathbb{C}$  cts  $\Rightarrow$  can uniformly approximate  $f$  by representative fns  
 (given  $\varepsilon > 0$ ,  $\exists \rho: G \rightarrow \text{Aut}(V)$ ,  $\exists \varphi \in \text{Hom}_{\mathbb{C}}(V, V)$  with  $\sup_{g \in G} |f - \text{Tr}(\varphi \circ \rho)| < \varepsilon$ )

Fact  $(f: G \rightarrow \mathbb{C}) \in \mathcal{C}(G) \Rightarrow$  can choose  $\varphi \in \text{Hom}_G(V, V) \subseteq \text{Hom}_{\mathbb{C}}(V, V)$

Stone-Weierstrass theorem (NON-EXAMINABLE)

- $M$  compact mfd (more generally a compact Hausdorff topological space)
- $S \subseteq C(M) = \{ \text{cts } M \rightarrow \mathbb{C} \}$  is \*-subalgebra separating points with 1  $\in S$   
 Then  $S \subseteq C(M)$  is dense!  
 $f \in S \Rightarrow \overline{f} \in S \Rightarrow S \subseteq C(M)$  subring and vector subspace  $\uparrow m \neq m' \Rightarrow \exists f \in S$  with  $f(m) \neq f(m')$

Rmk also holds for  $\text{IF} = \mathbb{R}$  so  $S \subseteq C(M, \mathbb{R})$ .

EXAMPLE  $S = \text{Span}_{\mathbb{C}} \{ f_a(x) = e^{2\pi i ax}: a \in \mathbb{Z} \} \subseteq C(S^1)$  where  $x \in S^1 = \mathbb{R}/\mathbb{Z}$

$S$  is v.s. since it's a span, and a subalgebra since  $f_a f_b = f_{a+b} \in S$   
 $f_a = e^{-2\pi i ax} = f_{-a} \in S$        $x \neq y \pmod{\mathbb{Z}} \Rightarrow e^{2\pi i x} \neq e^{2\pi i y}$   
 $\Rightarrow S = \{ \text{"trig polynomials"} \} \subseteq C(S^1)$  dense!

EXAMPLE for  $\text{IF} = \mathbb{R}$  use  $\text{Span}_{\mathbb{R}} \{ \cos 2\pi ax, \sin 2\pi ax: a \in \mathbb{N} \} \subseteq C(S^1, \mathbb{R})$ .

## CONSEQUENCE: FOURIER ANALYSIS

$$\|f\|_{L^2}^2 = \int_S |f|^2 d\mu < \infty$$

Recall  $C(S) \subseteq L^2(S) = \{\text{square-integrable fns } f: S \rightarrow \mathbb{C}\}$  dense (easy fact)  
 $\Rightarrow$  can  $L^2$ -approximate any  $L^2$ -function by trig polys. We knew this:

- $f = \left( \text{Fourier series } \sum_{a \in \mathbb{Z}} z_a e^{2\pi i ax} \right) \approx \sum_{a=-N}^N z_a f_a \text{ for } N \gg 0.$
- $z_a = \int_0^1 e^{-2\pi i ax} f(x) dx = \int_0^1 \overline{f_a(x)} f(x) dx = \langle f_a, f \rangle \in \mathbb{C}$   $\left( \begin{array}{l} z_a = \frac{1}{2\pi} \int_0^{2\pi} e^{-ia\theta} f(\theta) d\theta \\ \text{if parametrize by } \theta \in [0, 2\pi] \end{array} \right)$

## PROOF OF PETER-WEYL FOR MATRIX GROUP

CLAIM  $G = \text{compact matrix group} \Rightarrow F(G) \subseteq C(G)$  dense

Pf Already showed  $S = F(G)$  is  $\mathbb{C}$ -algebra.

I  $\in S$  since = (1,1) entry of trivial rep  $G \rightarrow \text{Aut}(\mathbb{C})$ ,  $g \mapsto \text{Id}$ .

$f = L \circ \rho \in S \Rightarrow \overline{f} = L \circ \bar{\rho} \in S$  using dual rep  $V^* \cong \overline{V}$  (Q.sheet 5,  $G$  compact)

Separate points: use standard rep  $G \subseteq GL(n, \mathbb{R})$  acting on  $\mathbb{C}^n$  by matrix mult,  
 so  $\varphi: G \subseteq GL(n, \mathbb{R}) \longrightarrow GL(n, \mathbb{C})$  inclusion, and if  $g_1 \neq g_2$  are different  
 matrices then some entry must be different  $\Rightarrow f(g_1) \neq f(g_2)$  some  $f = \text{Tr}(\varphi \circ \rho) \in S$  ■

## COMPACT LIE GPS ARE MATRIX GROUPS

### Peter-Weyl theorem (version 2)

$G$  compact Lie gp  $\Rightarrow \exists$  faithful rep  $G \longrightarrow U(m)$  some  $m$   
 $(= \text{injective})$

We proved version 2  $\Rightarrow$  version 1, Q.sheet 6: version 1  $\Rightarrow$  version 2!

## REGULAR REP, FOURIER ANALYSIS, $\infty$ -DIM REPS (NON-EXAMINABLE!)

### Peter-Weyl Theorem (version 3)

$\downarrow$  (v.s. with inner product, complete)

Any unitary rep  $G \rightarrow U(H) \subseteq \text{Aut}(H)$  on a Hilbert space  $H$   
 is an orthogonal direct sum of finite dim'l unitary subreps

$$H = \widehat{\bigoplus} W_i \quad \leftarrow \text{(allow infinite sums if cgt in } H \text{ i.e. closure of the usual direct sum)}$$

Recall for finite gp  $G$  the regular rep is  $H = \text{Functions}(G, \mathbb{C})$ .  
 $H$  is a v.s. of  $\dim = |G|$  with basis  $e_h$  indexed by  $h \in G$  given by:

$$e_h = (\text{function } h \mapsto 1 \text{ and all other } g \mapsto 0)$$

G-action:  $g \cdot e_h = e_{gh}$  (since on functions  $(g \cdot e_h)(gh) = e_h(g^{-1}gh) = 1$ )  
 $\Rightarrow G \xrightarrow{\text{faithful}} \{\text{permutation matrices}\} \subseteq \text{Aut}(\mathbb{C}^{|G|})$ .

FACT  $G$  finite  $\Rightarrow$  reg. rep.  $H \cong \bigoplus (\dim V_i) \cdot V_i$  summing over all irreps

So in principle can find all irreps of  $G$ !

Can't work for compact Lie  $G$  unless allow  $\infty$ -dim reps since for infinite  $G$  (compact) there are countably infinitely many finite dim'l irreps (for example for  $SU(2)$ , Q.sheet 5). This will follow from PW version 4 below.

Regular rep  $H = L^2(G) = \{\text{square-integrable } f : G \rightarrow \mathbb{C}\}$

Hilbert space with  $\langle f_1, f_2 \rangle = \int_{g \in G} \overline{f_1(g)} f_2(g) d\mu(g)$  and  $G$ -action:  $(h \cdot f)(g) = f(h^{-1}g)$

Peter-Weyl Theorem (version 4)  $L^2(G) = \bigoplus W_i$

where  $W_i \simeq (\dim V_i) \cdot V_i$  and all finite-dim irreps  $V_i$  arise

FACT An orthonormal analysis-basis for  $L^2(G)$  is  $\sqrt{\dim V_i} \cdot p_i^{(jk)}$  where  $p_i^{(jk)}(g) = \langle p(g) \cdot v_j, v_k \rangle$  is the  $(j,k)$  matrix entry in o.n. basis  $v_j$  for  $V_i$ .  
Analysis-basis  $e_n$  means linear combos can be infinite convergent sums so each  $v \in H$  is uniquely  $v = \sum_{n \in \mathbb{Z}} z_n e_n$  ( $\dim L^2(G)$  countable Q.sheet 6)

EXAMPLE  $L^2(S')$

Reg. rep.  $\rho : G \rightarrow \text{Aut}(L^2(S'))$ ,  $(\rho(x) \cdot f)(y) = f(y-x)$  ( $x, y \in S' = \mathbb{R}/\mathbb{Z}$ )

Claim  $L^2(S') = \bigoplus_{a \in \mathbb{Z}} \mathbb{C} \cdot e^{-2\pi i ax}$  (if parametrize  $S'$  by  $e^{i\theta}$   
then  $(\rho(e^{i\theta}) f)(e^{i\varphi}) = f(e^{-i\theta} e^{i\varphi}) = f(e^{i(\varphi-\theta)})$ )

Pf By P.W. thm:  $L^2(S') = \bigoplus_{a \in \mathbb{Z}} V_a$  (irreps of  $S'$  have  $\dim=1$ ,  $X_a = e^{2\pi i ax}$ )

hence  $V_a = \{f \in L^2(S') : \rho(x) \cdot f = e^{2\pi i ax} \cdot f\}$  (put  $y=0$  replace  $x$  by  $-x$ )  
 $= \mathbb{C} \cdot e^{-2\pi i ax}$  ■ (f(y-x) =  $e^{2\pi i ax} f(y)$ )  
(f(x) = f(0)  $\cdot e^{-2\pi i ax}$ )

Rmk  $F(G) \subseteq C(G) \subseteq L^2(G)$  are dense inclusions  
 $\text{Span}_{\mathbb{C}}(X) \subseteq C(G)^G = C(G)^G \subseteq L^2(G)^G$  are dense inclusions. ← by Thm 1  
What is character of  $L^2(S')$ ? (conjugation invariant  $L^2$ -functions)

$\chi_{L^2(S')} = \sum_{a \in \mathbb{Z}} e^{2\pi i ax}$  does not converge to a function.

It does however converge to a distribution (i.e. linear functional  $C^\infty(S') \rightarrow \mathbb{C}$ )

$f \in C^\infty(S') \mapsto \int \sum_{a=-N}^N e^{2\pi i ax} f(x) dx = \sum_{a=-N}^N z_{-a} \rightarrow \sum_{a \in \mathbb{Z}} z_a = f(0)$   
 $f \mapsto f(0)$  is the delta-function!  $\sum z_a e^{2\pi i ax} = f(x)$  at  $x=0$  Fourier coeffs

General G:  $\infty$ -dim rep  $\rho$ , the character is distribution  $\chi_H(f) = \text{Tr} \left( \int_{g \in G} f(g) \rho(g) \right)$

For reg. rep  $L^2(G)$ :  $\chi_{L^2(G)}(f) = f(1)$  is delta-function at 1

⇒ recover  $f$ :  $\chi_{L^2(G)}(f \circ \phi_h) = f(\phi_h(1)) = f(h)!$