

Over $\mathbb{F} = \mathbb{C}$

- G compact \Rightarrow Reps decompose $\rho = \bigoplus n_i \rho_i$ into irreps
- G abelian \Rightarrow Irreps are 1-dimensional, $G \rightarrow GL(1, \mathbb{C}) = \mathbb{C} \setminus \{0\}$
- G compact & abelian \Rightarrow Irreps must land in $S^1 \subseteq \mathbb{C} \setminus \{0\}$

Proof If $\rho(g) = \lambda \in \mathbb{C} \setminus \{0\}$ with $|\lambda| > 1$ then $\rho(g^n) = \lambda^n \rightarrow \infty$
 " " " " " $|\lambda| < 1$ then $\rho(g^{-n}) = \lambda^{-n} \rightarrow \infty$

But $\text{Image } (\rho : G \rightarrow \mathbb{C} \setminus \{0\})$ is compact hence bounded, so $|\lambda| = 1$ ■

Hence:

$$\begin{array}{ccc} \bullet G \text{ compact \& abelian} \Rightarrow & \left\{ \text{Irreps} \right\} & \xleftrightarrow{\substack{\text{1:1} \\ \text{equivalence}}} \left\{ \begin{array}{c} \text{Lie gp homs} \\ G \rightarrow S^1 \end{array} \right\} \\ & \rho = \chi_\rho \cdot \text{Id} & \longleftrightarrow \chi_\rho \end{array}$$

$\mathbb{F} = \mathbb{C}, G = S^1$ We know Lie gp homs $G = S^1 \rightarrow S^1$ so irreps are:

$$\rho_a : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow GL(1, \mathbb{C}), x \mapsto e^{2\pi i a x} \cdot \text{Id} \quad (a \in \mathbb{Z})$$

$\mathbb{F} = \mathbb{C}, G = T^n$ We know Lie gp homs $G = T^n \rightarrow S^1$ so irreps are:

$$\rho_a : T^n = \mathbb{R}^n/\mathbb{Z}^n \rightarrow GL(1, \mathbb{C}), \rho_a(x) = e^{2\pi i \langle a, x \rangle} \cdot \text{Id} \quad \begin{pmatrix} a \in \mathbb{Z}^n \\ \langle a, x \rangle = a_1 x_1 + \dots + a_n x_n \end{pmatrix}$$

 $\mathbb{F} = \mathbb{R}$

G compact \Rightarrow Reps decompose $\rho = \bigoplus n_i \rho_i$ into irreps

G abelian \Rightarrow Irreps have $\dim_{\mathbb{R}} = 1$ or 2

Proof $G \rightarrow GL(n, \mathbb{R}) \subseteq GL(n, \mathbb{C})$

so as a complex rep, it has a complex subrep $\mathbb{C} \cdot v \xrightarrow{v \in \mathbb{C}^n}$ of $\dim_{\mathbb{C}} = 1$

Let $x = \text{Re}(v) = \frac{1}{2}(v + \bar{v})$ and $y = \text{Im}(v) = \frac{1}{2i}(v - \bar{v})$ then

$\text{span}_{\mathbb{R}}(x, y) = \text{Re}(\mathbb{C}v) + \text{Im}(\mathbb{C}v) \subseteq \mathbb{R}^n$ is real subrep of $\dim_{\mathbb{R}} = 2$ or 1

Indeed: for $A = \rho(g)$: $Ax = \frac{1}{2}(Av + A\bar{v}) = \frac{1}{2}(Av + \overline{Av}) = \text{Re}(Av)$

similarly $Ay = \text{Im}(Av) \in \text{Im}(\mathbb{C}v)$. ■ $\xrightarrow{\text{A real}}$ $\in \mathbb{C}v$

$\mathbb{F} = \mathbb{R}, G = S^1$ Nontrivial irreps have $\dim_{\mathbb{R}} = 2$

Proof $\dim_{\mathbb{R}} = 1 \Rightarrow \dim_{\mathbb{C}} = 1$ irrep $\Rightarrow \rho = \chi_a \cdot \text{Id} \notin GL(1, \mathbb{R})$ ■

Claim $\rho_a^{\mathbb{R}} : G = S^1 \rightarrow GL(2, \mathbb{R})$, $\rho_a^{\mathbb{R}}(x) = \begin{pmatrix} \cos 2\pi a x & -\sin 2\pi a x \\ \sin 2\pi a x & \cos 2\pi a x \end{pmatrix}$ is irreducible. ($a \neq 0 \in \mathbb{Z}$)

Pf 1 $v \neq 0 \in \mathbb{R}^2 \Rightarrow v, \rho_a^{\mathbb{R}}\left(\frac{1}{4a}\right)v = (v \text{ rotated by } \frac{\pi}{2})$ are lin. indep, $\xrightarrow{\text{IR}}$

\Rightarrow no subreps except $0, \mathbb{R}^2$ ■ $\xrightarrow{\text{not real!}}$

Pf 2 A $\dim_{\mathbb{R}} = 1$ subrep would be a common eigenspace of all $\rho_a^{\mathbb{R}}(x)$ ■

Claim $\rho_a^{\text{IR}} \cong \rho_b^{\text{IR}} \iff a = -b$

Pf " \Rightarrow ": $\chi_{\rho_a}(x) = 2 \cos(2\pi ax)$ and recall $\rho \cong \rho' \Rightarrow \chi_\rho = \chi_{\rho'}$.
 " \Leftarrow ": $s^{-1} \circ \rho_a \circ s = \rho_{-a}$ if s = reflection in x -axis ■

Claim The irreps_{IR} of S^1 are $\begin{cases} \rho_a^{\text{IR}} \text{ for } a=1,2,3,\dots \in \mathbb{N} \\ \text{trivial irrep } S^1 \rightarrow \text{GL}(1, \mathbb{R}), x \mapsto \text{Id} \end{cases}$

Pf 1 ρ irrep of $\dim_{\mathbb{R}} = 2 \Rightarrow$ irrep over \mathbb{C} of $\dim_{\mathbb{C}} = 2 \Rightarrow \rho \cong \rho_a \oplus \rho_b$ over \mathbb{C}
 ρ real $\Rightarrow \rho = \bar{\rho} \Rightarrow \rho_a \oplus \rho_b \cong \bar{\rho}_a \oplus \bar{\rho}_b = \rho_{-a} \oplus \rho_{-b}$
 But $a = -a, b = -b$ would imply $a = 0, b = 0$ so $\rho = \text{trivial}$. So $a = -b$.
 $\Rightarrow \rho \cong \rho_a \oplus \rho_{-a}$. So \exists subrep $\mathbb{C}\nu, \nu \in \mathbb{C}^2$, with $\rho(x)\nu = e^{2\pi i ax}\nu$
 Then $\rho(x)\bar{\nu} = \overline{\rho(x)\nu} = \overline{e^{2\pi i ax}\nu} = e^{-2\pi i ax}\bar{\nu}$ ($\Rightarrow \mathbb{C}\bar{\nu}$ = subrep ρ_{-a})

Let $\alpha = \frac{1}{2i}(\nu - \bar{\nu}) = \text{Im}(\nu)$, $\beta = \frac{1}{2}(\nu + \bar{\nu}) = \text{Re}(\nu) \in \mathbb{R}^2$. As above, for $A = \rho(x)$:
 $A\alpha = \text{Im}(A\nu) = \text{Im}(e^{2\pi i ax}\nu) = \cos(2\pi ax) \cdot \text{Im}(\nu) + \sin(2\pi ax) \cdot \text{Re}(\nu)$
 $A\beta = \text{Re}(A\nu) = \text{Re}(e^{2\pi i ax}\nu) = \cos(2\pi ax) \cdot \text{Re}(\nu) - \sin(2\pi ax) \cdot \text{Im}(\nu)$
 so in basis $\alpha, \beta \in \mathbb{R}^2$ get $A = \rho(x) = \begin{pmatrix} \cos 2\pi ax & -\sin 2\pi ax \\ \sin 2\pi ax & \cos 2\pi ax \end{pmatrix} = \rho_a^{\text{IR}}(x)$. ■

Pf 2 Pick a G -invariant inner product on $\mathbb{R}^n \Rightarrow$ (up to an equivalence since change to an orthonormal basis of G -invt i.p.) can assume

$\rho: G = S^1 \rightarrow O(2) \subseteq \text{GL}(2, \mathbb{R})$ indeed land in $SO(2)$ since S^1 connected
 \Rightarrow if identify $\mathbb{R}^2 = \mathbb{C}$, (rotation by θ on \mathbb{R}^2) = (multiplication by $e^{i\theta}$ on \mathbb{C})
 \Rightarrow get $\dim_{\mathbb{C}} = 1$ \times irrep $\underset{(\text{some } a \in \mathbb{Z})}{\rho(x) = \rho_a(x) = e^{2\pi i ax}}$. $\text{Id}_{\mathbb{C}} = \begin{pmatrix} \cos 2\pi ax & -\sin 2\pi ax \\ \sin 2\pi ax & \cos 2\pi ax \end{pmatrix} = \rho_a^{\text{IR}}(x)$

Pf 3 (using Peter-Weyl) matrix entries of ρ_a are $\cos(2\pi ax), \sin(2\pi ax)$
 As vary $a \in \mathbb{N}$, these generate a dense subalgebra of $C(S^1, \mathbb{R})$ (Fourier analysis)
 The matrix entries of two non-equivalent irreps are orthogonal, so if ρ is an irrep different from ρ_a , then $\langle \chi_\rho, f \rangle = 0$ for $f \in$ (that subalgebra). But such f are dense in $C(S^1)$ so $\langle \chi_\rho, f \rangle = 0$ all $f \in C(S^1)$ so $\chi_\rho = 0$ contradicting the orthogonality relation $\langle \chi_\rho, \chi_\rho \rangle \geq 1$ (over \mathbb{R}) ■

Real irreps of T^n : similar argument, get trivial irrep \mathbb{R} and 2-dim irreps:

$$\rho_a: T^n \rightarrow \text{GL}(2, \mathbb{R}), \rho_a(x) = \begin{pmatrix} \cos 2\pi \langle a, x \rangle & -\sin 2\pi \langle a, x \rangle \\ \sin 2\pi \langle a, x \rangle & \cos 2\pi \langle a, x \rangle \end{pmatrix}$$

for $a \in \mathbb{Z}^n \setminus \{0\}$ and $\rho_a \cong \rho_b \iff a = -b$.