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Homework 5.

You are encouraged to collaborate on these exercises.

Question 1. Let H be a connected Lie group. Show that any discrete normal subgroup $N \subset H$ satisfies $N \subset \text{Centre}(H)$. *(Try it first, only then see the footnote for a hint.)*[1]

Let $\pi : H \to G$ be a covering of Lie groups, with H, G connected. Show that $\Gamma = \ker \pi$ is a discrete normal subgroup of $\text{Centre}(H)$.

Conversely, if $\Gamma \subset \text{Centre}(H)$ discrete, show[2] that H/Γ is a Lie group and that the quotient map $\pi : H \to H/\Gamma$ is a covering map with fibre $\ker \pi = \Gamma$.

Deduce that any connected Lie group with Lie algebra $\mathfrak{g}$ is isomorphic to G/Γ for some discrete subgroup $\Gamma \subset \text{Centre}(G)$, where G is a simply-connected Lie group.

Question 2. Let $\rho_j : G \to GL(d_j, \mathbb{F})$ be representations, $j = 1, 2$. State in terms of matrices what the following representations are: $\rho_1 \oplus \rho_2$, $\rho_1 \otimes_{\mathbb{F}} \rho_2$, conjugate rep $\overline{\rho_1}$, dual rep ρ_1^*, and $\text{Hom}_{\mathbb{F}}(\rho_1, \rho_2)$.

For compact G, show that $V^* \cong \overline{V}$. (*Hint. inner product.*)

Question 3. For V a representation (more precisely, $\rho : G \to \text{Aut}(V)$), define its **character** $\chi_V = \chi_\rho$ by

$$\chi_V : G \to \mathbb{F}, \quad \chi_V(g) = \text{Trace}(\rho(g)).$$

Check the following properties hold:

(1) χ_V is smooth
(2) $\chi_V(1) = \dim_{\mathbb{F}} V$
(3) χ_V is invariant under conjugation, $\chi_V(hgh^{-1}) = \chi_V(g)$
(4) $\chi_V = \chi_W$ for equivalent reps $V \simeq W$
(5) $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$
(6) $\chi_{V \otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$
(7) $\chi_{V^*}(g) = \chi_V(g^{-1})$
(8) $\chi_{\overline{V}}(g) = \overline{\chi_V(g)}$

Question 4. For G compact, and $\mathbb{F} = \mathbb{C}$, check the $1:1$ correspondence:

$$\{\text{1-dim reps}\}/\text{equivalence} \quad \overset{1:1}{\leftrightarrow} \quad \{\text{Lie group homs } G \to S^1\}, \quad \rho \mapsto \chi_\rho.$$

Classify all representations of S^1 and of T^n for $\mathbb{F} = \mathbb{C}$.

Observe that real representations $\rho : G \to \text{Aut}(\mathbb{R}^n)$ are also complex representations $\rho : G \to \text{Aut}(\mathbb{C}^n)$ satisfying $\rho(g) = \overline{\rho(g)}$ for all g. Suppose, in this situation, that $\mathbb{C}v$ is a 1-dim complex G-submodule of $\mathbb{C}^n$. Check that $x = \text{Re}(v) = \frac{1}{2}(v + \overline{v})$ and $y = \text{Im}(v) = \frac{1}{2i}(v - \overline{v})$ span a 2-dim real G-submodule of $\mathbb{R}^n$.

Then classify all representations of S^1 and of T^n for $\mathbb{F} = \mathbb{R}$. *(See the footnote for hints.)*[3]

Question 5. Canonical decomposition. For compact G, and $\mathbb{F} = \mathbb{C}$, and V_i the (inequivalent) irreducible reps of G, show that the following evaluation map is a G-isomorphism:

$$\mathbf{ev} : \bigoplus_i \text{Hom}_G(V_i, V) \otimes_{\mathbb{F}} V_i \to V,$$

where on a generator $\varphi \otimes v$ we define $\mathbf{ev}(\varphi \otimes v) = \varphi(v)$, and then extend $\mathbf{ev}$ linearly.

[1] Hint. Recall the definition of Centre from Question sheet 4. The results from Q. sheet 4 don't help here. Instead, let γ_t be a path from 1 to h, then observe that for $n \in N$ the continuous path $\gamma_t n \gamma_t^{-1}$ lies in N. But N is discrete.

[2] Hint: easier than it looks, combine results from Lectures 8 and 10. Hint to prove that Γ is closed: suppose $g_m \in \Gamma$ are distinct with $g_m \to g \in H$, then $g_m^{-1} g_{m+1} \to 1 \in \Gamma$ using the continuous map $H \times H \to H, (h, g) \mapsto h^{-1}g$.

[3] Hints: recall Q.2 on Question sheet 3 classifies Lie group homs $T^n \to S^1$. In $\mathbb{R}^2$, if s is a reflection in the x-axis and r is a rotation by θ, check that $s^{-1} \circ r \circ s$ is a rotation by $-\theta$. Use Q.3.(4) of this sheet to distinguish some of the irreps.