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Homework 1.

You are encouraged to collaborate on these exercises.

Question 1. Show that the *tangent bundle* $TG = \bigsqcup_{g\in G} T_gG$ of a Lie group G is canonically identifiable with $G \times T_IG$.
Hint. consider the left translation map $\phi_g : G \to G$, $\phi_g(h) = gh$.

Deduce that any Lie group of dimension n has n non-vanishing vector fields which are linearly independent at each point of G.

Deduce that the 2-dimensional sphere S^2 *cannot* be a Lie group.
Hint. you may quote the "hairy ball theorem" – google it!

Show that the 3-dimensional sphere S^3 is a Lie group by considering

$$SU(2) = \{2 \times 2 \text{ complex matrices with } A^\dagger A = I,\ \det A = 1\}$$

where $A^\dagger$ denotes the conjugate transpose of A.
Hint. Verify that $SU(2)$ *is the set of matrices* $\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a}\end{smallmatrix}\right)$ *with* $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$.

Cultural remark. The only spheres which are also Lie groups are S^0, S^1, S^3.

Question 2. Suppose G_1, G_2 are Lie groups. Show that $G_1 \times G_2$ is a Lie group in a natural way.

Deduce that the n-dimensional torus $T^n = S^1 \times \cdots \times S^1$ is a Lie group.

Find a map $\pi : \mathbb{R}^n \to T^n$ that allows you to identify $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$ (the quotient group).

Not all vector fields on $\mathbb{R}^n$ give rise to vector fields on T^n if you apply $D\pi$, but which ones do? Are these all the vector fields on T^n?

Find out which vector fields on T^n are *left-invariant*, meaning

$$D_h\phi_g \cdot X|_h = X|_{gh}$$

for all $h, g \in G$, where ϕ_g is defined in Question 1.

Question 3. Use the implicit function theorem (at the end of Lecture 1) applied to

$$\varphi : GL(n, \mathbb{R}) \to \mathrm{Sym}(n, \mathbb{R}) = \{n \times n \text{ symmetric matrices }\},\ \varphi(A) = A^TA,$$

to prove that the orthogonal group $O(n)$ is a Lie group, to find the dimension of $O(n)$ and to find the tangent space $T_IO(n)$.

Show that $O(n)$ is compact. *Hint. You may quote the Heine-Borel theorem.*

Question 4. Let $\varphi : M \to N$ be a *diffeomorphism* of manifolds (a smooth map with smooth inverse). For a vector field X on M define the *push-forward* vector field $Z = \varphi_*X$ on N by

$$Z|_y = D_x\varphi \cdot X|_x$$

where $x = \varphi^{-1}(y)$. Show that for any function $f : N \to \mathbb{R}$,

$$(\varphi_*X) \cdot f = (X \cdot (f \circ \varphi)) \circ \varphi^{-1}.$$

Deduce that $[\varphi_*X, \varphi_*Y] \cdot f = \varphi_*[X, Y] \cdot f$, and deduce that

$$[\varphi_*X, \varphi_*Y] = \varphi_*[X, Y].$$

Check that this last identity holds in the simple case: $M = N = \mathbb{R}$, $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial x}$, $\varphi(x) = 2x$.

Let G be a Lie group. Prove the following characterization of left-invariant vector fields:

$$X \in \mathrm{Lie}\, G \ \Leftrightarrow\ (\phi_g)_*X = X \quad \text{for all } g \in G,$$

and deduce that, if $X, Y \in \mathrm{Lie}\, G$, then also $[X, Y] \in \mathrm{Lie}\, G$.

Remark. It's tricky to show $[\varphi_*X, \varphi_*Y] = \varphi_*[X, Y]$ *directly using coordinates, try it if you are brave.*

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Homework 2.

You are encouraged to collaborate on these exercises.

Question 1. Viewing quaternions as matrices, show that quaternions satisfy the rules

$$|h_1 h_2| = |h_1| \cdot |h_2| \qquad |h^{-1}| = |h|^{-1}.$$

Viewing $\mathbb{H}$ as a real 4-dimensional vector space, check that $|h|$ is the usual norm on $\mathbb{R}^4$.
Show that *(using Lecture 2 and Question sheet 1)*

$$\mathrm{Sp}(1) = SU(2).$$

For $h \in \mathbb{H} \setminus \{0\}$ define

$$\mathcal{A}_h \colon \mathbb{H} \to \mathbb{H}, p \mapsto hph^{-1}.$$

Show that $\mathcal{A}_h$ is an orthogonal map (viewing $\mathbb{H}$ as $\mathbb{R}^4$). *(Hint. recall Example 11 from Lecture 2.)*
By considering the orthogonal complement of $\mathbb{R} = \mathbb{R} \cdot 1 \subset \mathbb{H}$, deduce that $\mathrm{SU}(2) \cong \mathrm{Sp}(1) \subset \mathbb{H} \setminus \{0\}$ acts on $\mathbb{R}^3$ by rotations.
Writing quaternions as $r + v$, where $r \in \mathbb{R}$ and $v \in \mathbb{R}^3 = \mathrm{span}_{\mathbb{R}}(i, j, k)$, show that

$$v_1 v_2 = -v_1 \bullet v_2 + v_1 \times v_2$$

for $v_1, v_2 \in \mathbb{R}^3$, where $\bullet$ is dot product in $\mathbb{R}^3$, and $\times$ is cross product in $\mathbb{R}^3$.
Show that any $h \in \mathrm{Sp}(1)$ can be written as

$$h = \cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2}) v$$

for a unit vector $v \in \mathbb{R}^3$ and for some $\theta \in \mathbb{R}$. Show that in this case $vv = -1$ and $\mathcal{A}_h(v) = v$.
Describe the rotation determined by h. *(Hint. consider an orthonormal basis $w_1, w_2, v \in \mathbb{R}^3$.)*
Deduce that there is a smooth surjective homomorphism

$$\mathrm{SU}(2) \to \mathrm{SO}(3)$$

and explain briefly in what sense SU(2) "covers" SO(3) twice.
Show that SO(3) as a manifold is a solid ball $B^3 \subset \mathbb{R}^3$ of radius π having identified the antipodal points on the boundary of the ball (this boundary is a sphere of radius π in $\mathbb{R}^3$). This space is called *real projective space*, $\mathbb{R}P^3$.
Taking inspiration from the construction of polar coordinates, show that $\mathbb{R}P^3$ can be identified with the space of straight lines in $\mathbb{R}^4$ through the origin. Finally, show that the map SU(2) $\to$ SO(3) corresponds to the map

$$S^3 \to \mathbb{R}P^3, \quad (x \in S^3 \subset \mathbb{R}^4) \mapsto (\text{the straight line in } \mathbb{R}^4 \text{ through the two points } 0 \text{ and } x).$$

Question 2. Check these properties of $\exp : \mathrm{Lie}(G) \to G$.

(1) $\mathrm{Image}(\exp) \subset G_0 =$ connected component of $1 \in G$;
(2) $\exp((t+s)v) = \exp(tv)\exp(sv)$ for all $t, s \in \mathbb{R}$;
(3) $(\exp v)^{-1} = \exp(-v)$;
(4) If $g = \exp(v)$ then it has an n-th root: $\exp(\frac{1}{n}v)$;
(5) Show that the following map is not surjective

$$\exp : \mathfrak{sl}(2, \mathbb{R}) \to SL(2, \mathbb{R})$$

by considering the eigenvalues of the square root (if it existed) of $g = \begin{pmatrix} -4 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$.

Cultural remark. *For any compact connected Lie group G,* exp *is surjective.*

Question 3. *Remark. Abbreviate* $\mathfrak{g} = \mathrm{Lie}(G)$*. By Lecture 5 you know that* $\mathbf{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *is a Lie algebra homomorphism because it is the derivative* $D_1\mathrm{Ad}$ *of a Lie group homomorphism.*

Prove directly that $\mathbf{ad}$ is a Lie algebra homomorphism by using the fact that $\mathbf{ad}(X) \cdot Z = [X, Z]$.

Show that

$$v_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

is a basis for $\mathfrak{so}(3) \subset \mathrm{Mat}_{3\times 3}(\mathbb{R})$.

By computing all brackets $[v_i, v_j]$, show that

$$\mathfrak{so}(3) \cong (\mathbb{R}^3, \text{cross product}), \ v_i \mapsto \text{standard basis vector } e_i$$

is a Lie algebra isomorphism.

Via this isomorphism we identify $\mathrm{End}(\mathfrak{so}(3))$ with 3×3 matrices. Compute the matrices $\mathbf{ad}(v_i)$.

By computing $\langle v_i, v_j \rangle$ show that the **Killing form**

$$\langle v, w \rangle = \mathrm{Trace}(\mathbf{ad}(v)\mathbf{ad}(w)) \in \mathbb{R}$$

is a negative definite scalar product on $\mathfrak{so}(3)$.

Remark. Observe

$$\mathfrak{su}(2) \cong \mathfrak{so}(3) \cong \mathfrak{o}(3),$$

since $SU(2)$, $SO(3)$, $O(3)$ are locally diffeomorphic near 1.

Cultural Remark. *for a compact Lie group, the Killing form is negative definite on* $\mathfrak{g}/\ker \mathbf{ad}$ *(here we quotiented by the centre* $Z(\mathfrak{g}) = \mathrm{Lie}(Z(G)) = \ker \mathbf{ad}$ *because the Killing form is zero if* $\mathbf{ad}(v) = 0$*).*

Homework 3.

You are encouraged to collaborate on these exercises.

Question 1. Show that the subgroups of $S^1 = \mathbb{R}/\mathbb{Z}$ are: S^1 or one of two types:

(1) a finite subgroup generated by a rational number;
(2) an infinite subgroup which is dense in S^1.

Describe geometrically the 1-parameter sugroups of the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. In particular, give an example of a subgroup $\mathbb{R} \subset T^2$ which is not a submanifold.[1]

Question 2. Let $\varphi : T^n \to S^1$ be a Lie group homomorphism. Show that $D_1\varphi$ has integer entries. *(Hint. use naturality of* exp*, and try the case $n = 1$ first if you get stuck.)*

Determine all Lie group homomorphisms

$$\varphi : T^n \to S^1$$

and all Lie group homomorphisms

$$T^n \to T^n.$$

(Hint. given $D_1\varphi \in \mathbb{Z}^n$, can you construct a homomorphism φ? is it unique?)

Let $v \in \mathbb{R}^n$. If the subgroup $\langle v \rangle$ generated by v is not dense in $T^n = \mathbb{R}^n/\mathbb{Z}^n$, show that $v \in \ker(\varphi : T^n \to S^1)$ for some non-trivial φ.

(Hint. what Lie group can $T^n/\overline{\langle v \rangle}$ be, using the final results of Lecture 6?)

Show that the following statements are equivalent for $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$:

(1) $1, v_1, \ldots, v_n$ are linearly dependent over $\mathbb{Q}$;
(2) $\sum a_i v_i \in \mathbb{Z}$ for some $a_i \in \mathbb{Z}$, where not all a_i are zero;
(3) $\langle v \rangle$ is not dense in T^n.

Deduce that almost any $v \in T^n$ will generate a dense subset of T^n!

Question 3. Using the formulas from Lecture 5, obtain the formula

$$\boxed{\exp(X)\exp(Y)\exp(-X) = \exp(Y + [X,Y] + \tfrac{1}{2!}[X,[X,Y]] + \tfrac{1}{3!}[X,[X,[X,Y]]] + \cdots)}$$

Show that for a matrix group,

$$\mathrm{Ad}(g) \cdot X = gXg^{-1}$$

where $g \in G, X \in \mathfrak{g}$.

Consider the subgroup $T \subset U(n)$ of diagonal unitary matrices. Show that T is a torus and that T lies in the image of $\exp : \mathfrak{u}(n) \to U(n)$. Deduce that

$$\exp : \mathfrak{u}(n) \to U(n)$$

is surjective.

(Hint. Recall from linear algebra that a unitary matrix has a basis of unitary eigenvectors.)

Question 4. Suppose

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k$$

as a vector space. Let

$$\mathfrak{g} \to G, \quad \psi(v_1, \ldots, v_k) = \exp(v_1) \cdots \exp(v_k).$$

Show that[2]

$$D_0\psi \cdot (X_1, \ldots, X_k) = X_1 + \cdots + X_k,$$

and deduce that ψ is a local diffeomorphism near 0.

[1] $N \subset M$ is a submanifold if the inclusion is an embedding, i.e. a homeomorphism onto the image (in the subspace topology) and the derivative of the inclusion is injective.

[2] where we naturally identify $T_0\mathfrak{g} = \mathfrak{g}$, [curve $0 + tX$] $\leftrightarrow X$.

Therefore, for small $X, Y \in \mathfrak{g}$, we can uniquely define $f(X,Y) \in \mathfrak{g}$ by the equation

$$\exp X \cdot \exp Y = \exp(f(X,Y)).$$

Intuitively $f(X,Y)$ is telling you what group multiplication in G looks like in $\mathfrak{g}$ via $\log = (\exp)^{-1}$.
By Taylor[3] expanding f near $(0,0)$, show that there is a bilinear map $B : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$ such that

$$f(X,Y) = X + Y + \tfrac{1}{2}B(X,Y) + \text{higher order terms.}$$

Using $\exp(Z)^{-1} = \exp(-Z)$, show that B is antisymmetric. Using the formula of Q.3, show

$$B(X,Y) = [X,Y].$$

Cultural Remark.

$$f(X,Y) = \exp^{-1}(\exp(X)\exp(Y)) = X + Y + \tfrac{1}{2}[X,Y] + \text{higher}$$

*is called the **Baker-Campbell-Hausdorff formula**. A hard theorem states that the higher order terms can all be expressed in terms of Lie brackets involving X and Y (see Wikipedia). This proves the remarkable fact that the local group structure of G (multiplication for elements near 1) is determined by the Lie algebra $\mathfrak{g}$.*

[3]Recall Taylor says: $f(X,Y) = f(0,0) + D_0(f) \cdot \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) + \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)^T \cdot \text{Hessian}_0(f) \cdot \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) + \cdots$. To ensure that the Hessian term does not have x_i^2, y_i^2 terms, consider $f(X,0)$ and $f(0,Y)$.

Homework 4.

You are encouraged to collaborate on these exercises.

Question 1. Let $\varphi : G_1 \to G_2$ be a Lie group homomorphism. Show that

$$\ker \varphi \subset G_1$$

is a closed (hence embedded) Lie subgroup with Lie algebra

$$\ker(D_1\varphi) \subset \mathfrak{g}_1.$$

*A vector subspace $J \subset (V, [\cdot,\cdot])$ of a Lie algebra is called an **ideal** if*

$$[v, j] \in J \text{ for all } v \in V, j \in J.$$

Show that ideals are Lie subalgebras. Show that for a Lie subgroup $H \subset G$, with H, G connected,

$$\boxed{H \subset G \text{ is a normal subgroup} \Leftrightarrow \mathfrak{h} \subset \mathfrak{g} \text{ is an ideal}}$$

Hints. for $\Leftarrow$ use the formula from Question 1. For $\Rightarrow$ use that formula but put tX, sY instead of X, Y and show that the curve $e^{t\,\mathrm{ad}\,X} \cdot Y$ lies in $\mathfrak{h}$.

*The **centre** of a Lie algebra $(V, [\cdot,\cdot])$ is*

$$Z(V) = \{v \in V : [v, w] = 0 \text{ for all } w \in V\}.$$

For G connected, prove that the centre of the group G is[1]

$$\boxed{Z(G) = \ker(\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}))}$$

Deduce that the centre of G is a closed (hence embedded) Lie subgroup of G which is abelian, normal and has Lie algebra

$$\mathrm{Lie}(Z(G)) = Z(\mathfrak{g}).$$

Finally deduce that, for G connected,

$$\boxed{G \text{ is abelian} \Leftrightarrow \mathfrak{g} \text{ is abelian}}$$

Question 2. Show that

$$[X, Y] = 0 \Rightarrow \exp(X + Y) = \exp(X)\exp(Y).$$

(Hint. By Lecture 8, Lie subalgs of $\mathfrak{g}$ correspond to connected Lie subgps of G. Consider span(X, Y).)

Prove that if G is a Lie group **with $\mathbf{Z(G) = \{1\}}$** then G can be identified with a Lie subgroup of $GL(m, \mathbb{R})$, some m, so $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(m, \mathbb{R})$.

If $(V, [\cdot,\cdot])$ is a Lie algebra **with $\mathbf{Z(V) = \{0\}}$**, show that V is the Lie algebra of some Lie group. *(Hint. consider $\mathbf{ad} : V \to \mathrm{End}(V), \mathbf{ad}(X) \cdot Y = [X, Y]$, and use the theorem in the previous hint.)*

Cultural remark 1. *A big theorem (**Ado's theorem**) states that any Lie algebra V has a faithful representation into some $\mathfrak{gl}(m, \mathbb{R})$ (that is, an injective Lie algebra homomorphism $V \to \mathfrak{gl}(m, \mathbb{R})$). The same arguments you used above imply that there is a Lie subgroup of $GL(m, \mathbb{R})$ with Lie algebra V. So one could reduce the study of Lie algebras to studying matrices with the bracket $[B, C] = BC - CB$.*

Cultural remark 2. *Another big theorem (**Lie's third theorem**) states: if you impose the topological condition that the Lie group should be simply-connected[2] then you also get uniqueness:*

$$\{\text{ Lie algebras } V\}/\text{Lie alg isos} \overset{1:1}{\longleftrightarrow} \{\text{ connected simply-connected Lie groups } G\}/\text{Lie gp isos}$$

That condition is necessary, since the double cover $SU(2) \to SO(3)$ illustrates two different Lie groups with isomorphic Lie algebras (but only $SU(2)$ is simply connected).

[1]Recall the centre of a group is $Z(G) = \{g \in G : hg = gh \text{ for all } h \in G\} = \{g \in G : hgh^{-1} = g \text{ for all } h \in G\}$.

[2]meaning continuous loops can always be continuously deformed to a point.

All connected Lie groups having a given Lie algebra are obtained from the corresponding simply-connected Lie group by quotienting by a central discrete sugroup. In the example, $SO(3) = SU(2)/\{\pm I\}$.

Cultural Remark 3. *Not all Lie groups are matrix groups. The Heisenberg group*

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

is a simply-connected matrix group (as a manifold, it's just $\mathbb{R}^3$*), but it turns out that the quotient*

$$H / \begin{pmatrix} 1 & 0 & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

does not admit a faithful representation into any $\mathfrak{gl}(m, \mathbb{R})$.

Question 3. Find all the connected Lie subgroups of $SO(3)$.
Hint. Use the results from Q.3 of Question sheet 2.

Question 4. Given any real or complex matrix X, show that

$$\det e^X = e^{\mathrm{Tr}}(X).$$

(Hint. Recall from linear algebra, that over $\mathbb{C}$ *any matrix is conjugate to an upper triangular matrix.)*

Deduce that

$$\mathfrak{sl}(n, \mathbb{R}) = \{A \in \mathrm{Mat}_{n \times n}(\mathbb{R}) : \mathrm{Tr}(X) = 0\}.$$

Deduce that

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is a basis of $\mathfrak{sl}(2, \mathbb{R})$ and check that

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Why is the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ not isomorphic to $\mathfrak{so}(3)$?

Which connected Lie subgroup of $SL(2, \mathbb{R})$ corresponds to the Lie subalgebra $\mathbb{R} \cdot (f - e)$?

Which connected Lie subgroup of $SL(2, \mathbb{R})$ corresponds to the Lie subalgebra $\mathrm{span}(h, e)$?

A Lie group is called ***simple*** *if it is connected, non-abelian, and has no non-trivial connected normal subgroups. A Lie algebra* V *is called* ***simple*** *if it is non-abelian, and its only ideals are* 0 *and* V.

Prove in general the correspondence:

$$\{ \text{ connected normal subgroups of G } \} \stackrel{1:1}{\longleftrightarrow} \{ \text{ideals of } \mathfrak{g} \}$$

Deduce that a connected Lie group is simple if and only if its Lie algebra is simple.

By considering $\mathfrak{sl}(2, \mathbb{R})$, show that $SL(2, \mathbb{R})$ is a simple Lie group.

Question 5. Let V_n be the vector space of homogeneous[3] polynomials of degree n in two variables z_1, z_2. Show that $SU(2)$ acts on V_n by

$$(A \cdot p)(z) = p(zA),$$

where $p \in V_n$, $A \in SU(2)$, and zA is matrix multiplication of the row-vector $z = (z_1, z_2)$ with A. Deduce that the V_n are representations[4] of the Lie group $SU(2)$ of dimension $n + 1$.

Cultural Remark. *In fact, these are all the irreducible*[5] *representations of* $SU(2)$. *Here* V_0 *is the trivial representation,* V_1 *is the standard representation, and* V_n *is called the* n*-th symmetric power of* V_1.

By considering the double cover $SU(2) \to SO(3)$, and using the cultural remark, show that the irreducible representations of $SO(3)$ are precisely the spaces V_{2n} of odd dimension $2n + 1$.

---

[3] meaning: the total degree of each term is the same, for example $3z_1^2 + 4z_1z_2 - 5z_2^2$ is homogeneous of degree 2.

[4] Recall a representation R of a group G is a vector space R together with a Lie group homomorphism $\varphi : G \to \mathrm{Aut}(R)$.

[5] Irreducible means that the only vector subspaces $R' \subset R$ satisfying $g \cdot R' \subset R'$, for all $g \in G$, are $R' = 0$ and $R' = R$ (recall we abbreviate $g \cdot r' = \varphi(g)(r')$).

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Homework 5.

You are encouraged to collaborate on these exercises.

Question 1. Let H be a connected Lie group. Show that any discrete normal subgroup $N \subset H$ satisfies $N \subset \mathrm{Centre}(H)$. *(Try it first, only then see the footnote for a hint.)*[1]

Let $\pi : H \to G$ be a covering of Lie groups, with H, G connected. Show that $\Gamma = \ker \pi$ is a discrete normal subgroup of $\mathrm{Centre}(H)$.

Conversely, if $\Gamma \subset \mathrm{Centre}(H)$ discrete, show[2] that H/Γ is a Lie group and that the quotient map $\pi : H \to H/\Gamma$ is a covering map with fibre $\ker \pi = \Gamma$.

Deduce that any connected Lie group with Lie algebra $\mathfrak{g}$ is isomorphic to G/Γ for some discrete subgroup $\Gamma \subset \mathrm{Centre}(G)$, where G is a simply-connected Lie group.

Question 2. Let $\rho_j : G \to GL(d_j, \mathbb{F})$ be representations, $j = 1, 2$. State in terms of matrices what the following representations are: $\rho_1 \oplus \rho_2$, $\rho_1 \otimes_{\mathbb{F}} \rho_2$, conjugate rep $\overline{\rho_1}$, dual rep ρ_1^*, and $\mathrm{Hom}_{\mathbb{F}}(\rho_1, \rho_2)$.

For compact G, show that $V^* \cong \overline{V}$. (*Hint. inner product.*)

Question 3. For V a representation (more precisely, $\rho : G \to \mathrm{Aut}(V)$), define its **character** $\chi_V = \chi_\rho$ by

$$\chi_V : G \to \mathbb{F}, \quad \chi_V(g) = \mathrm{Trace}(\rho(g)).$$

Check the following properties hold:

(1) χ_V is smooth
(2) $\chi_V(1) = \dim_{\mathbb{F}} V$
(3) χ_V is invariant under conjugation, $\chi_V(hgh^{-1}) = \chi_V(g)$
(4) $\chi_V = \chi_W$ for equivalent reps $V \simeq W$
(5) $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$
(6) $\chi_{V\otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$
(7) $\chi_{V^*}(g) = \chi_V(g^{-1})$
(8) $\chi_{\overline{V}}(g) = \overline{\chi_V(g)}$

Question 4. For G compact, and $\mathbb{F} = \mathbb{C}$, check the $1:1$ correspondence:

$$\{\text{1-dim reps}\}/\text{equivalence} \quad \overset{1:1}{\leftrightarrow} \quad \{\text{Lie group homs } G \to S^1\}, \quad \rho \mapsto \chi_\rho.$$

Classify all representations of S^1 and of T^n for $\mathbb{F} = \mathbb{C}$.

Observe that real representations $\rho : G \to \mathrm{Aut}(\mathbb{R}^n)$ are also complex representations $\rho : G \to \mathrm{Aut}(\mathbb{C}^n)$ satisfying $\rho(g) = \overline{\rho(g)}$ for all g. Suppose, in this situation, that $\mathbb{C}v$ is a 1-dim complex G-submodule of $\mathbb{C}^n$. Check that $x = \mathrm{Re}(v) = \frac{1}{2}(v + \overline{v})$ and $y = \mathrm{Im}(v) = \frac{1}{2i}(v - \overline{v})$ span a 2-dim real G-submodule of $\mathbb{R}^n$.

Then classify all representations of S^1 and of T^n for $\mathbb{F} = \mathbb{R}$. *(See the footnote for hints.)*[3]

Question 5. Canonical decomposition. For compact G, and $\mathbb{F} = \mathbb{C}$, and V_i the (inequivalent) irreducible reps of G, show that the following evaluation map is a G-isomorphism:

$$\mathbf{ev} : \bigoplus_i \mathrm{Hom}_G(V_i, V) \otimes_{\mathbb{F}} V_i \to V,$$

where on a generator $\varphi \otimes v$ we define $\mathbf{ev}(\varphi \otimes v) = \varphi(v)$, and then extend $\mathbf{ev}$ linearly.

---

[1]Hint. Recall the definition of Centre from Question sheet 4. The results from Q. sheet 4 don't help here. Instead, let γ_t be a path from 1 to h, then observe that for $n \in N$ the continuous path $\gamma_t n \gamma_t^{-1}$ lies in N. But N is discrete.

[2]Hint: easier than it looks, combine results from Lectures 8 and 10. Hint to prove that Γ is closed: suppose $g_m \in \Gamma$ are distinct with $g_m \to g \in H$, then $g_m^{-1} g_{m+1} \to 1 \in \Gamma$ using the continuous map $H \times H \to H, (h, g) \mapsto h^{-1}g$.

[3]Hints: recall Q.2 on Question sheet 3 classifies Lie group homs $T^n \to S^1$. In $\mathbb{R}^2$, if s is a reflection in the x-axis and r is a rotation by θ, check that $s^{-1} \circ r \circ s$ is a rotation by $-\theta$. Use Q.3.(4) of this sheet to distinguish some of the irreps.

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Homework 6. - Do collaborate...

All Lie groups are assumed compact, and we work over $\mathbb{F} = \mathbb{C}$.
We'll prove some harder theorems on this sheet. Use the footnotes to guide you through the argument.

Question 1. Irreducibility criterion: prove that V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.
If V_1, V_2 are irreps of G_1, G_2, show $V_1 \otimes V_2$ is an irrep of $G_1 \times G_2$.
Claim:[1] Conversely, all irreps of $G_1 \times G_2$ have the form $V_1 \otimes V_2$, for irreps V_1, V_2 of G_1, G_2 respectively.

Question 2. Representation theory for SU(2) *Recall (Q.5 Sheet 4) $SU(2)$ acts by $(A \cdot p)(z) = p(zA)$ on $p \in V_n = \{$homogeneous degree n polys in z_1, z_2 over $\mathbb{C}\}$. We'll use the basis $P_j = z_1^j z_2^{n-j}$, $0 \le j \le n$.*
Claim 1.[2] The V_n are irreducible.
Claim 2.[3] The characters χ_n of the V_n are uniformly dense in $\mathrm{Cl}(SU(2))$.
Claim 3.[4] The V_n are the only irreps of $SU(2)$ (up to equivalence).

Question 3. Claim.[5] **Every compact Lie group admits a faithful rep into some U(n).**
Remark. $U(n) \to SO(2n)$ embeds via $A \mapsto \left(\begin{smallmatrix} \mathrm{Re}\, A & -\mathrm{Im}\, A \\ \mathrm{Im}\, A & \mathrm{Re}\, A \end{smallmatrix}\right)$, so we can replace $U(n)$ by $O(n)$ above.

Question 4. Claim 1.[6] The span over $\mathbb{C}$ of the image of $\chi : R(G) \to \mathrm{Cl}(G)$ is dense, that is: class functions f can be uniformly approximated by $\sum z_i \chi_{V_i}$ for $z_i \in \mathbb{C}$.
Claim 2.[7] The matrix entries of a faithful representation $\rho : G \to U(n)$, together with the conjugates of the entries, and with 1, generate the $\mathbb{C}$-algebra $\mathcal{F}(G)$ of all representative functions.
Claim 3.[8] Every irrep of G is a subrep of $V^{\otimes a} \otimes \overline{V}^{\otimes b}$, some $a, b \in \mathbb{N}$, where $V = \mathbb{C}^n$ is the faithful rep $\rho : G \to U(n)$. *Remark. This implies that $L^2(G)$ has countable dimension (see Lecture 13).*
Claim 4.[9] For a closed (so compact Lie) subgp $H \subset G$, any irrep of H is contained inside an irrep of G.

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[1] Let V be a rep of $G_1 \times G_2$. Then V is a rep of $G_2 = 1 \times G_2 \subset G_1 \times G_2$. Apply the canonical decomposition (Q.5 Sheet 5) to V, G_2. Define a G_1-action on $\mathrm{Hom}_{G_2}(V_2, V)$ for an irrep V_2 of G_2 so that the decomposition becomes $G_1 \times G_2$-linear. Apply complete reducibility to the G_1-mod $\mathrm{Hom}_{G_2}(V_2, V)$.

[2] By the irreducibility criterion of Q.1, V_n is irrep iff $\mathrm{Hom}_{SU(2)}(V_n, V_n) = 1$. So given $\varphi : V_n \to V_n$ $SU(2)$-linear map, need show $\varphi = c \cdot \mathrm{Id}$. Consider the diagonal matrices $D_\lambda \in SU(2)$ with entries λ, λ^{-1}. Compute the action of D_λ on P_j. Deduce that for $\lambda = e^{2\pi i/4n}$ the λ^{2j-n}-eigenspace of D_λ is spanned by P_j. Deduce that $\varphi(P_j) = c_j P_j$, some $c_j \in \mathbb{C}$. Consider the rotation $R_\theta \in SU(2)$ by θ. Expand $\varphi(R_\theta P_n) = R_\theta \varphi(P_n)$ to deduce that the c_j are all equal.

[3] Recall unitary matrices are diagonalizable. Deduce that any element in $SU(2)$ is conjugate to D_λ with $\lambda = e^{i\theta}$, uniquely up to changing θ to $-\theta$. Deduce that class functions $f : SU(2) \to \mathbb{C}$ are in $1:1$ correspondence with cts 2π-periodic even functions $\mathbb{R} \to \mathbb{C}$ via $\theta \mapsto f \circ D_{e^{i\theta}}$. So can abbreviate $\chi_n(D_{e^{i\theta}}) = \chi_n(\theta)$. Check $\chi_n(\theta) = \sum e^{i(2j-n)\theta}$. Compute that geometric sum, you should get $\chi_n(\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$, call this $c_n(\theta)$. Using trig identities, deduce $c_n = \cos(n\theta) + c_{n-1}(\theta)\cos\theta$. Deduce that the $\chi_n(\theta)$ generate (as an algebra) $1, \cos(\theta), \cos(2\theta), \ldots, \cos(n\theta)$. By basic Fourier analysis, <u>even</u> 2π-periodic continuous functions are uniformly approximated by $\cos(n\theta)$, $n \in \mathbb{N}$.

[4] Hint. Orthogonality relations and Claim 2.

[5] For a chain of strict inclusions $K_1 \supset K_2 \supset K_3 \supset \cdots$ of closed sub-manifolds, check that dim drops or the # of connected components drops each time. If K_j are closed subgps of G show the chain must stop. For a manifold M, and distinct points $p, q \in M$, explain why there is a cts function $f : M \to \mathbb{C}$ with $f(p) \neq f(q)$. Repeatedly apply this idea to $M = G$, using the Peter-Weyl theorem to approximate such f by representative functions $L \circ \rho$. The first step is to take $p = 1, q = g \neq 1$, to get $\rho_1 : G \to \mathrm{Aut}(V_1)$, with $K_1 = \ker \rho_1 \subset G$ a strict inclusion. You aim to end up with a faithful rep $V_1 \oplus \cdots \oplus V_m$.

[6] Use the canonical decomposition $\mathbf{ev} : \oplus H_i \otimes V_i \simeq V, \mathbf{ev}(\psi_i, v_i) = \psi_i(v_i)$ (Q.5 Sheet 5) where $H_i = \mathrm{Hom}_G(V_i, V)$, V_i the irreps. By Peter-Weyl $f \approx \mathrm{Tr}(\varphi \circ \rho)$, some $\varphi \in \mathrm{Hom}_G(V, V)$, $\rho : G \to \mathrm{Aut}(V)$. Check that $\varphi \circ \rho$ on V corresponds via the canonical decomposition to $\oplus(\varphi \otimes \rho_i)$ on $\oplus H_i \otimes V_i$. So the traces of those two maps agree. Final hint: $z_i = \mathrm{Tr}(\varphi \circ : H_i \to H_i)$.

[7] Let $\mathcal{M}(G)$ be the algebra generated. By Stone-Weierstrass show $\mathcal{M}(G) \subset C(G)$ is dense. Deduce $\mathcal{M}(G)$ is dense in $\mathcal{F}(G)$. Aim: $\mathcal{M}(G) \subset \mathcal{F}(G)$ closed in sup-norm. Now $\|f\|^2 = \langle f, f \rangle = \int_G \overline{f}(g) f(g) \le (\sup_G |f|)^2$, so sup-closure$(\mathcal{M}(G))$ is a subset of $\|\cdot\|$-closure$(\mathcal{M}(G))$. Deduce: if $\mathcal{M}(G)$ is $\|\cdot\|$-closed then both closures equal $\mathcal{M}(G)$. Recall $\mathcal{F}(G) = \oplus \mathcal{F}_{V_i}(G)$ is an orthogonal direct sum over irreps V_i of G. Orthogonal projection $\pi_i : \mathcal{F}(G) \to \mathcal{F}_{V_i}(G)$ satisfies $\|\pi_i(f - m)\| \le \|f - m\|$ for all $m \in \mathcal{M}(G)$. Deduce that for $f \in \overline{\mathcal{M}(G)}$ (the $\|\cdot\|$-closure), $\pi_i(f) \in \overline{\pi_i(\mathcal{M}(G)} \subset \mathcal{F}_{V_i}(G)$. Deduce, since $\dim \mathcal{F}_{V_i}(G) < \infty$, that $f \in \mathcal{M}(G)$.

[8] The matrix entries of $V^{\otimes a} \otimes \overline{V}^{\otimes b}$ are monomials of degree a in matrix entries of V and of degree b in matrix entries of $\overline{V}$. By Claim 2, they generate $\mathcal{F}(G)$, as a, b vary. If W were an irrep contradicting Claim 3, then by orthogonality $\int_G \overline{f_W} f_V = 0$ for all $f_W \in \mathcal{F}_W(G)$, $f_V \in \mathcal{F}_{V^{\otimes a} \otimes \overline{V}^{\otimes b}}(G)$. By Peter-Weyl and Claim 2 this is impossible.

[9] Apply Claim 3 to the faithful rep $H \to G \to \mathrm{Aut}(V)$.

OPTIONAL QUESTIONS (hand in if you like)

Optional Question 1. A Lie group that is not a matrix group. Consider

$$H = \{\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right) : a, b, c \in \mathbb{R}\} \qquad N = \{\left(\begin{smallmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}.$$

Check that $N \subset H$ is a closed normal subgroup, so H/N is a Lie group.

This is of interest in quantum mechanics, because H/N is faithfully represented by the group of operators $M_c S_b T_a$ (in that order!) acting on the Hilbert space $L^2(\mathbb{R})$, generated by translation $T_a f(x) = f(x+a)$, rescaling $S_b f(x) = e^{2\pi i b} f(x)$, and multiplication $M_c f(x) = e^{2\pi i c x} f(x)$. Check H/N is iso to this group.

Check that if you replace $n \in \mathbb{Z}$ by $n \in \mathbb{R}$ you obtain a circle $T \subset \mathrm{Centre}(H/N)$, and check[10] that each element in T is a commutator $ghg^{-1}h^{-1}$.

Prove in general that given a circle $T \subset \mathrm{Centre}(G)$, any rep of G is a sum $V = \oplus V_a$, such that T acts on V by $e^{2\pi i a x}$ where $a \in \mathbb{Z}$.

Prove in general that if elements of T can be written as commutators, then[11] in fact only V_0 is non-zero and therefore T is in the kernel of the representation.

Deduce that the Heisenberg group H/N **is not a matrix group**.

Optional Question 2. Real representations vs complex representations.

We saw in Q.4 sheet 5 that a real rep is also a complex rep via $G \to GL(n,\mathbb{R}) \subset GL(n,\mathbb{C})$ so $\rho(g) = \overline{\rho(g)}$. More abstractly, this is the process of **complexifying** a real rep W: we get $W_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} W$ with $\mathbb{C}$-action $\lambda(z \otimes w) = (\lambda z) \otimes w$ and G-action $g(z \otimes w) = z \otimes gw$.

Deduce that real reps arise as complex reps which are **self-conjugate**:[12] $\overline{V} \simeq V$.

Check that V is self-conjugate iff χ_V is real-valued.

In the reverse direction, a complex rep V gives a real rep $V_{\mathbb{R}}$: just consider V as a vector space over $\mathbb{R}$. Less abstractly: $\rho : G \to GL(n,\mathbb{C}) \subset GL(2n,\mathbb{R})$ where[13] $GL(n,\mathbb{C}) \subset GL(2n,\mathbb{R})$, $A \mapsto \left(\begin{smallmatrix} \mathrm{Re}\,A & -\mathrm{Im}\,A \\ \mathrm{Im}\,A & \mathrm{Re}\,A \end{smallmatrix}\right)$.

Prove that

$$(W_{\mathbb{C}})_{\mathbb{R}} \simeq W \oplus W \qquad\qquad (V_{\mathbb{R}})_{\mathbb{C}} \simeq V \oplus \overline{V}.$$

Let $R_{\mathbb{R}}(G) = \{\sum n_i W_i : n_i \in \mathbb{Z}, W_i \text{ real irreps of } G\}$ denote the real representation ring. Deduce that

$$R_{\mathbb{R}}(G) \to R(G), \quad W \mapsto \mathbb{C} \otimes_{\mathbb{R}} W$$

is an injective homomorphism. Hence, if W, W' are real reps with $\mathbb{C} \otimes_{\mathbb{R}} W \simeq \mathbb{C} \otimes_{\mathbb{R}} W'$ as cx reps, then $W \simeq W'$ as real reps. Therefore, once you've found out which self-conjugate cx reps are real reps, you just need to use this criterion to determine which are equivalent.

A good example to play with, which is an alternative approach to Q.4 Sheet 5, is to find the real irreps of S^1 given that we easily know the complex irreps (and similarly for real irreps of T^n).

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[10] *Hint.* $T_n M_1 T_n^{-1} M_1^{-1}$.

[11] *Hint.* consider determinants.

[12] Self-conjugate reps may fail to be real – they may be quaternionic, or a tensor of a real and a quaternionic rep.

[13] Even more explicitly, a complex basis $e_j^{\mathbb{C}}$ gives a real basis $e_1^{\mathbb{C}}, \ldots, e_n^{\mathbb{C}}, ie_1^{\mathbb{C}}, \ldots, ie_n^{\mathbb{C}}$. The first column of the inclusion is because $Ae_j^{\mathbb{C}} = (\mathrm{Re}\,A + i\mathrm{Im}\,A)e_j^{\mathbb{C}} = \mathrm{Re}\,(A)\,e_j^{\mathbb{C}} + \mathrm{Im}\,(A)\,ie_j^{\mathbb{C}}$. A simple example is:

$$e^{i\theta} \in GL(1,\mathbb{C}) \mapsto \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right) \in GL(2,\mathbb{R}).$$