A Tour of Tilings in Thirty Minutes

Alexander F. Ritter

Mathematical Institute & Wadham College
University of Oxford

Wadham College Mathematics Alumni Reunion
Oxford, 21 March, 2015.

For a detailed tour of Penrose tilings, see:
people.maths.ox.ac.uk/ritter/masterclasses.html

What is a tiling of the plane?

$\mathcal{M}$= tile-set of **model tiles**.

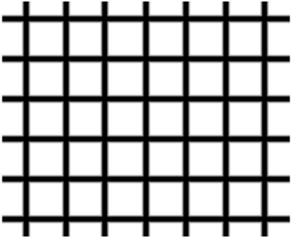
Tiling = a **covering** of the plane using model tiles such that

- **each point** of the plane lies **in some tile**,
- tiles **do not overlap** except along boundaries.

Examples: tiling by ...

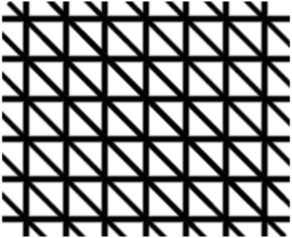
... squares

$\mathcal{M} = \{\square\}$



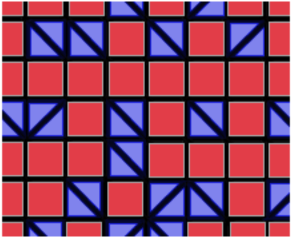
... isosceles triangles

$\mathcal{M} = \{\triangleright\}$

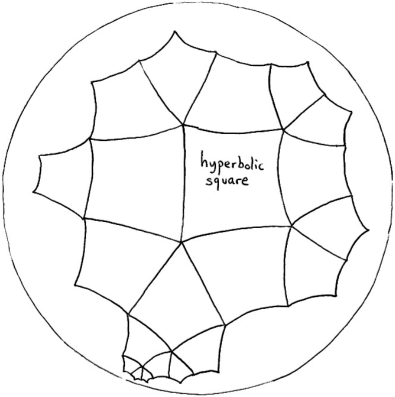


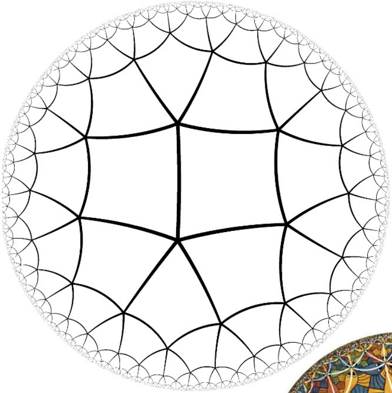
... a square and a triangle

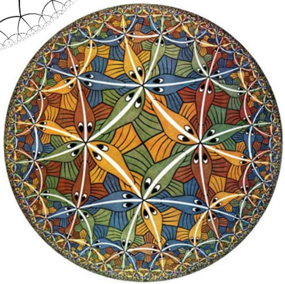
$\mathcal{M} = \{\blacksquare, \blacktriangle\}$



Tiling other spaces: the hyperbolic plane



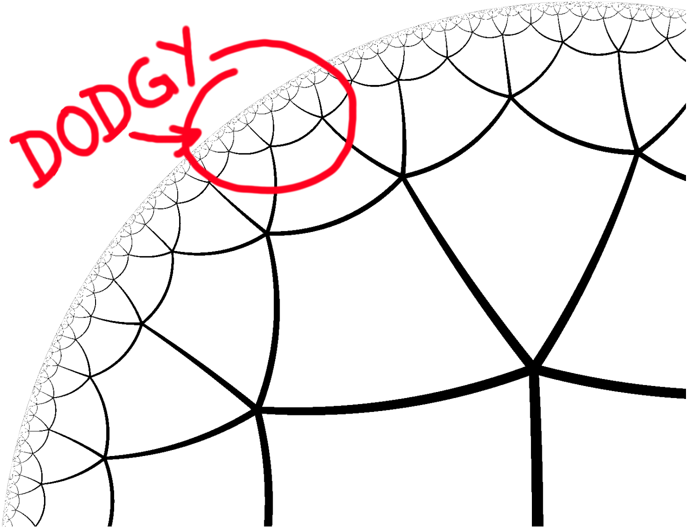




Can be used artistically:
M. C. Escher
Circle Limit III, 1959

I tile therefore I am (*Tessellatus ergo sum*)

Just because you've tiled a **small patch** of the plane, doesn't mean you can tile the **whole plane**! You may get stuck.

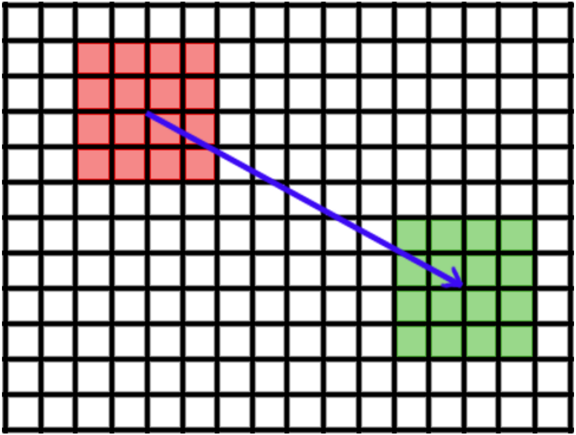


Translation symmetry

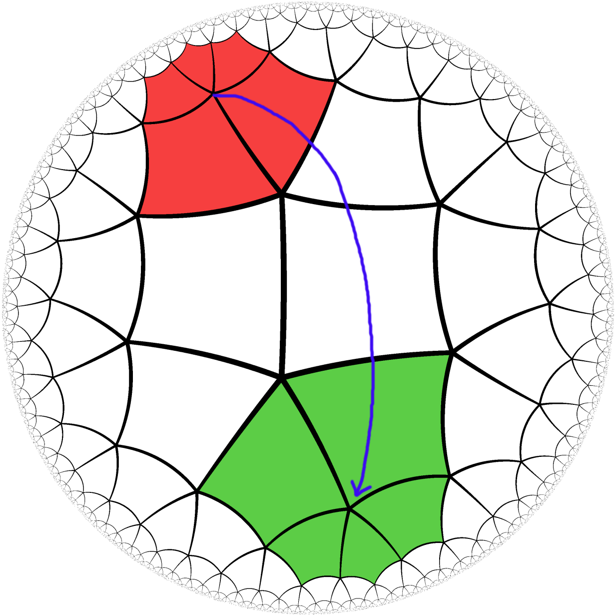
"Symmetry is what we see at a glance; based on the fact that there is no reason for any difference" [Pascal, *Pensées*, 60 Anno Wadhami]

Periodic tiling = if have translation symmetries in two directions.

Tile a big patch, then copy-paste with translations!



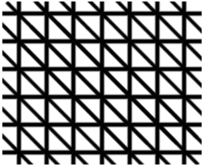
Also in the hyperbolic world, symmetries *prove* we don't get stuck

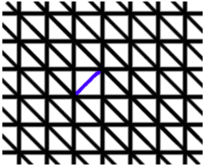


Non-repeating patterns

Non-periodic tiling $=$ if there is *no* translation symmetry.

Note: $\mathcal{M} = \{\triangleright\}$ admits both **periodic** and **non-periodic** tilings.





Aperiodic tile-set $=$ if $\mathcal{M}$ only admits non-periodic tilings.

A Brief History of Tiles:

1964 Robert Berger discovered an aperiodic tile-set: **20,426 tiles**.

1971 Raphael Robinson discovered an aperiodic tile-set: **6 tiles**.

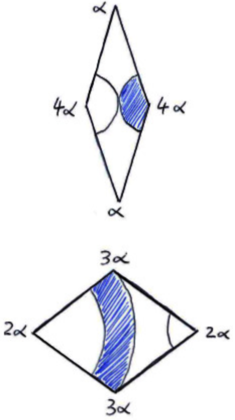
1974 Roger Penrose discovered an aperiodic tile-set: **2 tiles**.

I'm sorry Wadhamaticians, I'm afraid I can't do that [HAL9000]

If the input is $\mathcal{M}$, then **a computer cannot** output in finite time whether "**I can tile**" or "**I can't**" using $\mathcal{M}$. (Berger, 1964)

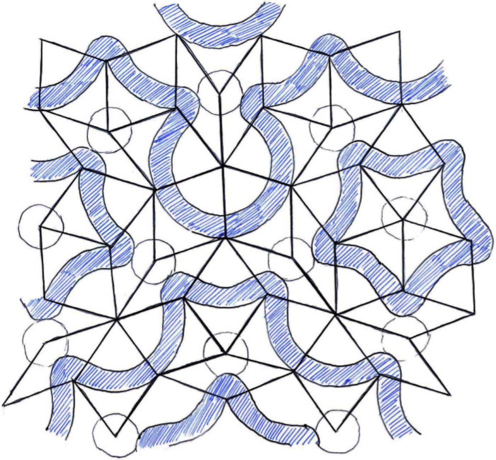
Penrose rhombi: an aperiodic tile-set

$\mathcal{M} = \{$



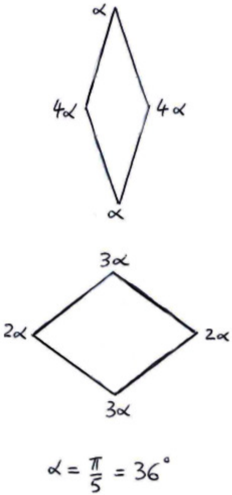
$\alpha = \frac{\pi}{5} = 36^\circ$

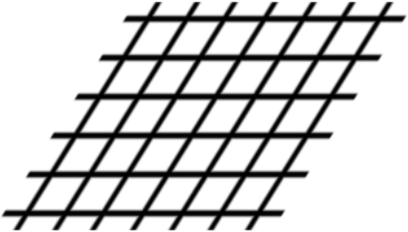
$\}$

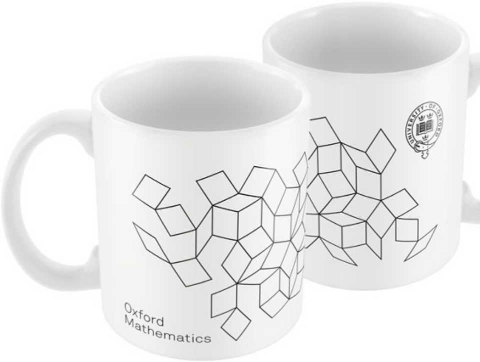


Why decorations on the tiles? It's not aperiodic otherwise

$\mathcal{M} =$

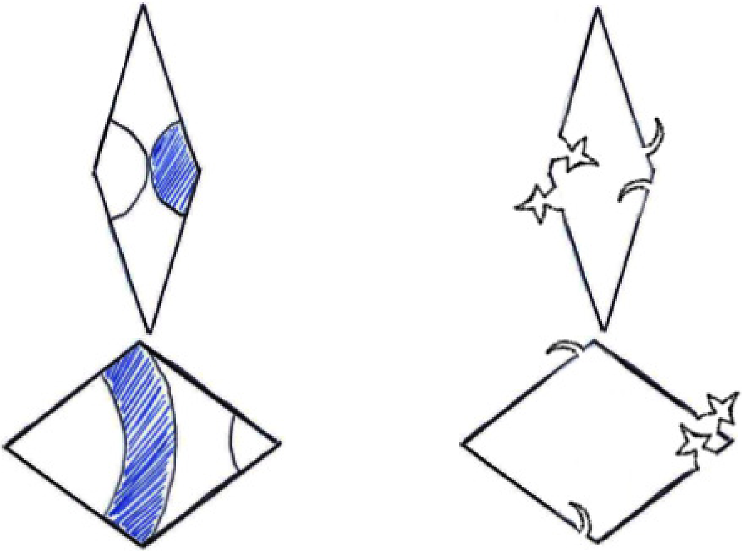






There is no permanent place in this world for ugly tilings

If we don't want decorations, we would need to put indentations:



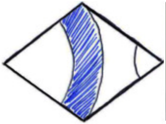
The two finest tilings in Oxford

Wadham College

Mathematical Institute

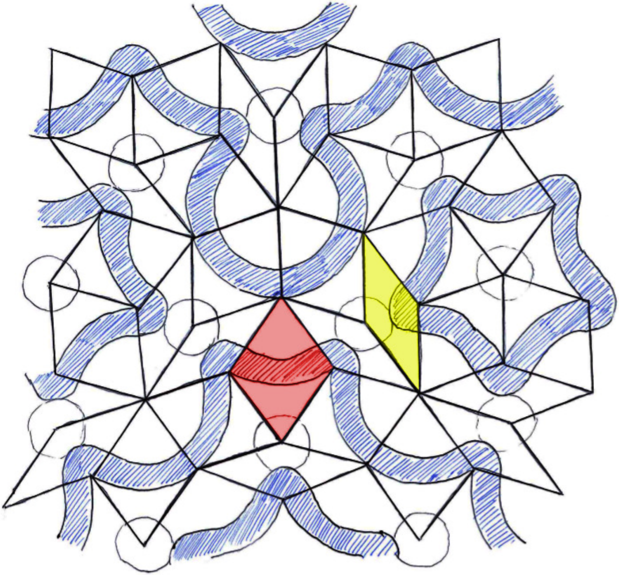
Oxford 2013

Wadham Guide to Penrose Tilings in 4 Steps

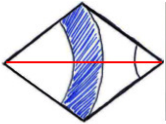



Dorothy Wadham

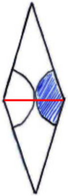

Nicholas Wadham



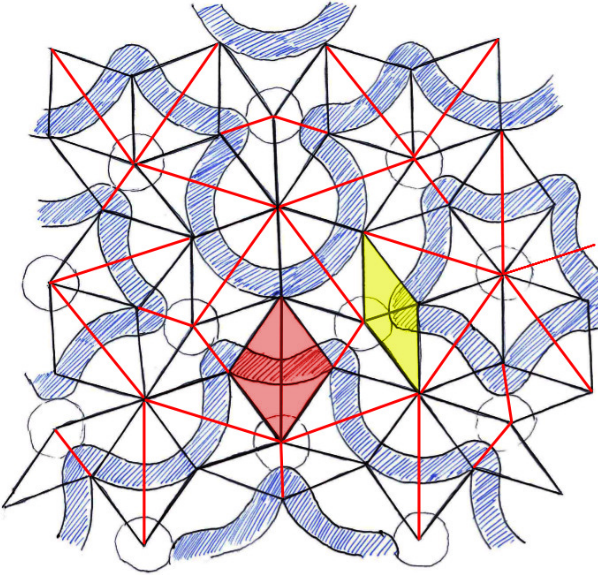
Step 1: Draw the diagonals



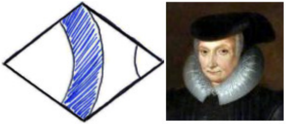
Dorothy Wadham



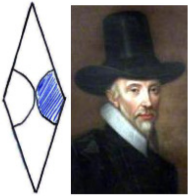
Nicholas Wadham



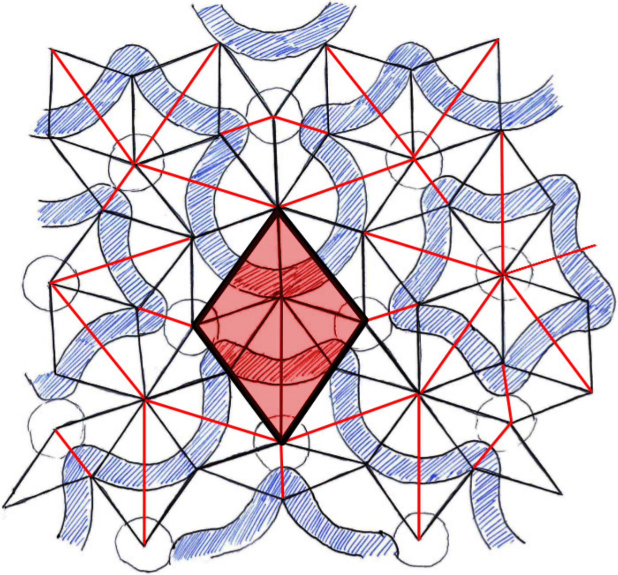
Step 2: Find Dorothy, but 60% larger (scaling: 1.618 . . .)



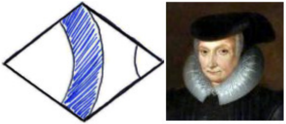
Dorothy Wadham



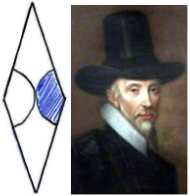
Nicholas Wadham



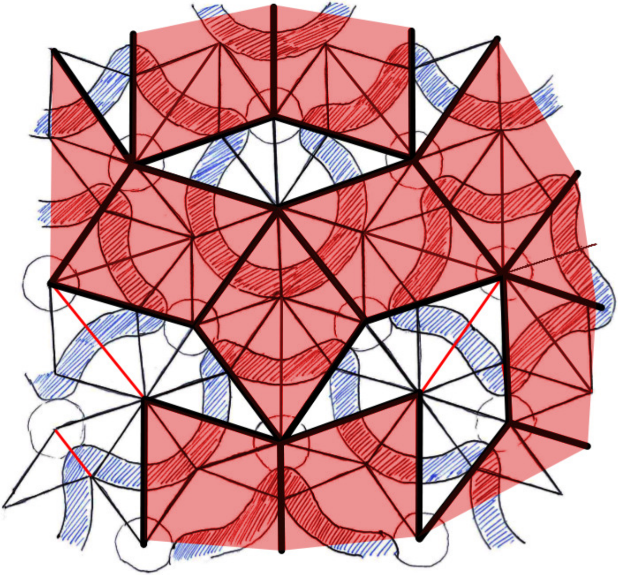
Step 3: Keep searching for Dorothy



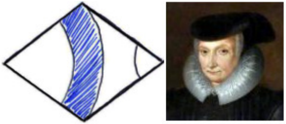
Dorothy Wadham



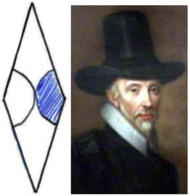
Nicholas Wadham



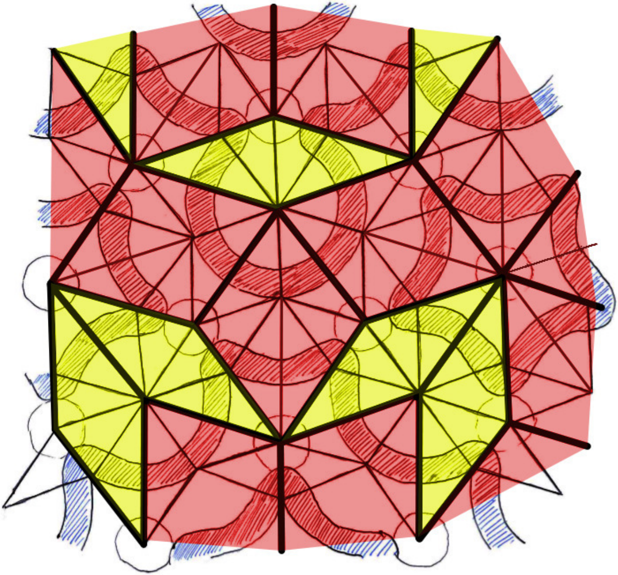
Step 4: Nicholas fills in the gaps



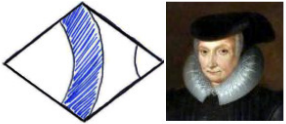
Dorothy Wadham



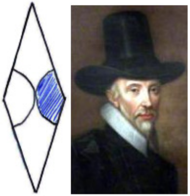
Nicholas Wadham



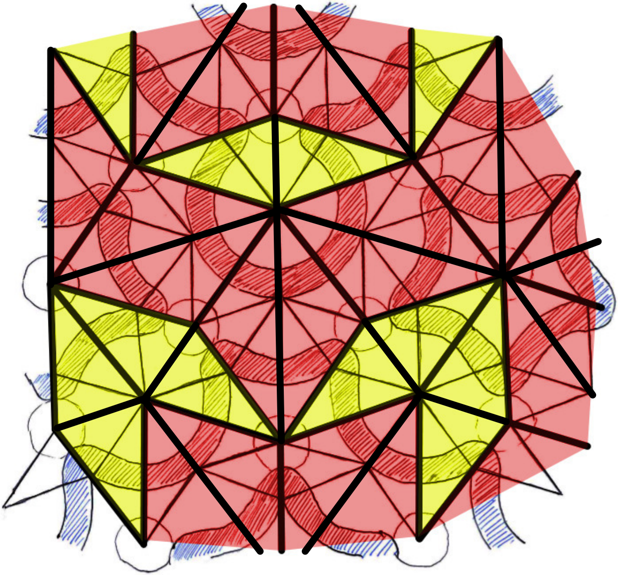
Step 1 (again): Draw the diagonals



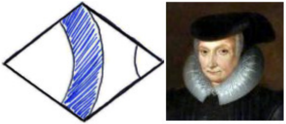
Dorothy Wadham



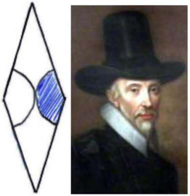
Nicholas Wadham



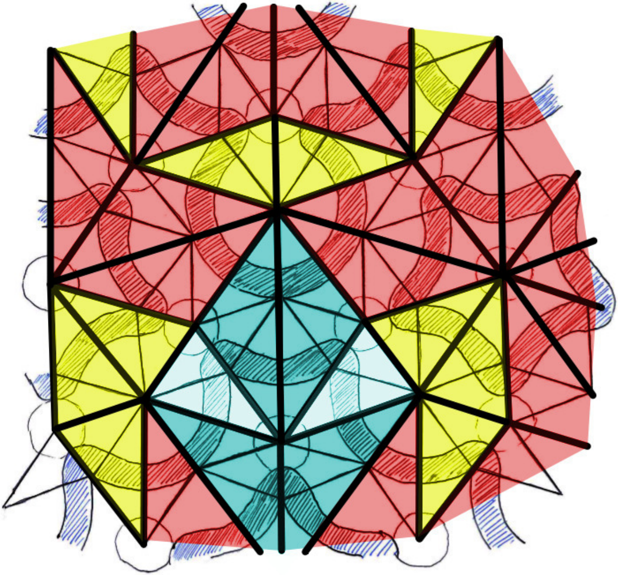
Step 2 (again): Find Dorothy, but 60% larger



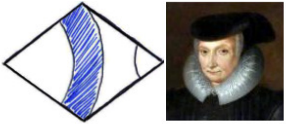
Dorothy Wadham



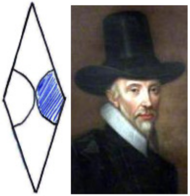
Nicholas Wadham



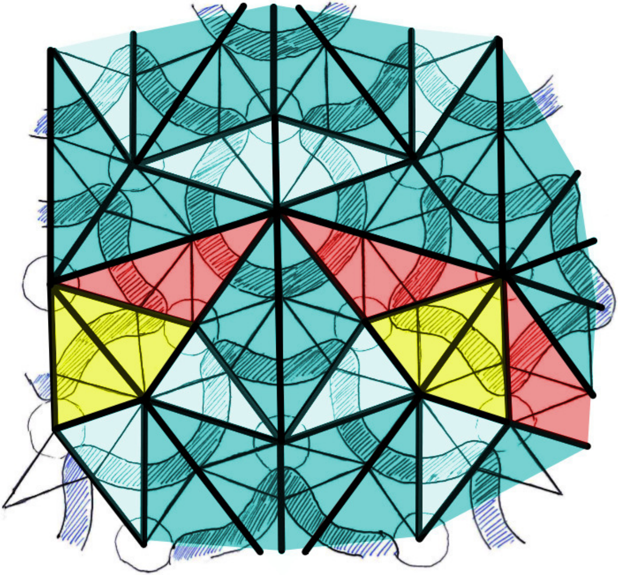
Step 3 (again): Keep searching for Dorothy



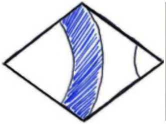
Dorothy Wadham



Nicholas Wadham

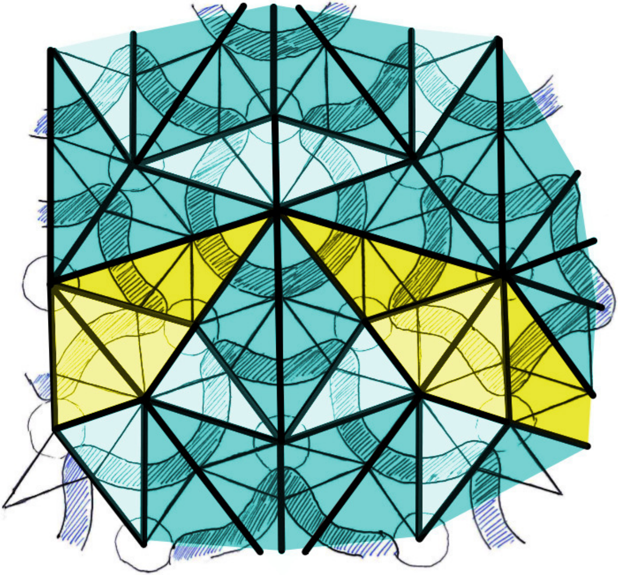


Step 4 (again): Nicholas fills in the gaps

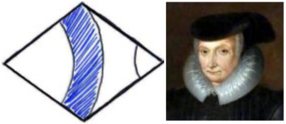


Dorothy Wadham

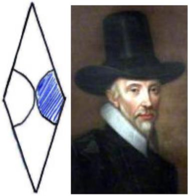
Nicholas Wadham



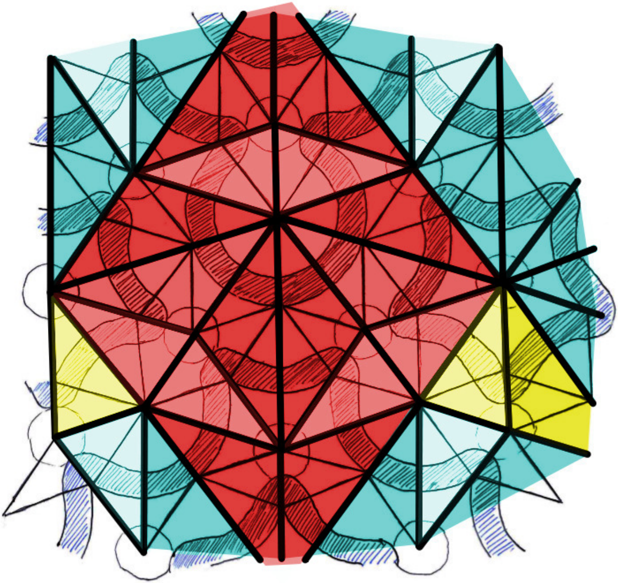
Repeat. For example we find the next Dorothy in Step 2:



Dorothy Wadham

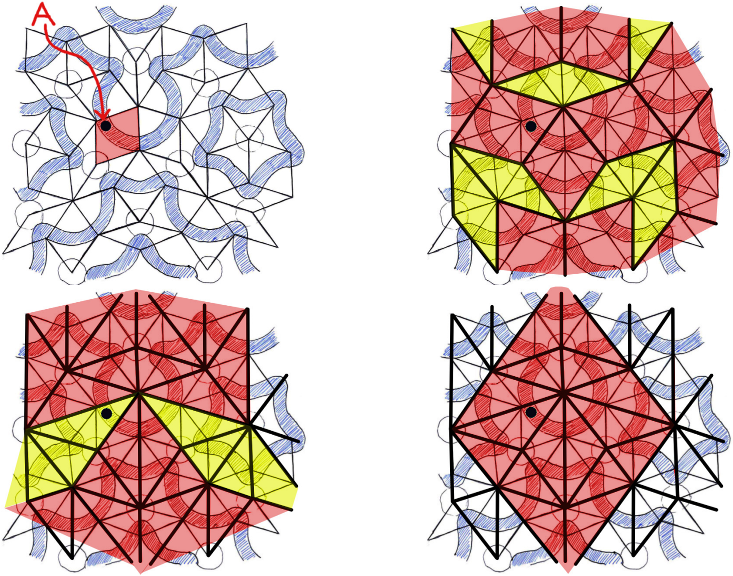


Nicholas Wadham



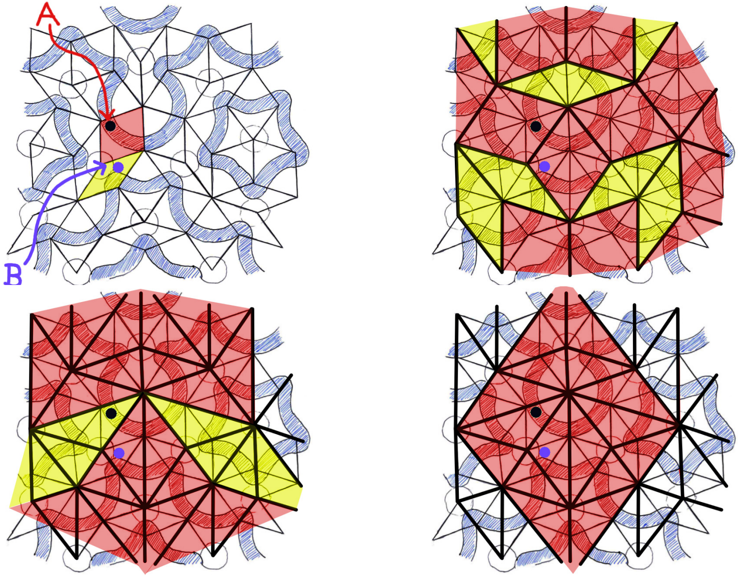
DNA sequence of a Penrose tiling

Pick a point A. Is A in a **D**-tile or an **N**-tile? Example: **DDND**...



For any two points, the DNA sequences eventually agree

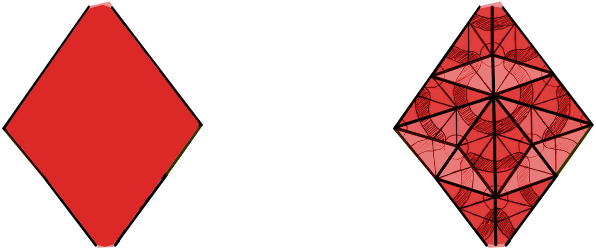
For the point A: **DDND******* For the point B: **NDDD*******



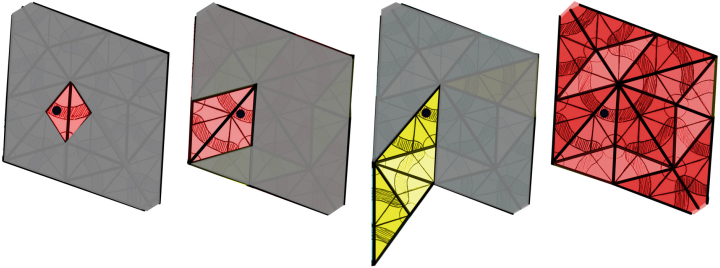
So DNA tells you whether two Penrose tilings are different!

From the DNA sequence, we can reconstruct the tiling

Key trick: there is a unique way to reverse Steps 1-4. Example:



From DNA can reconstruct tiling up to rotation/translation. For **DDND**:



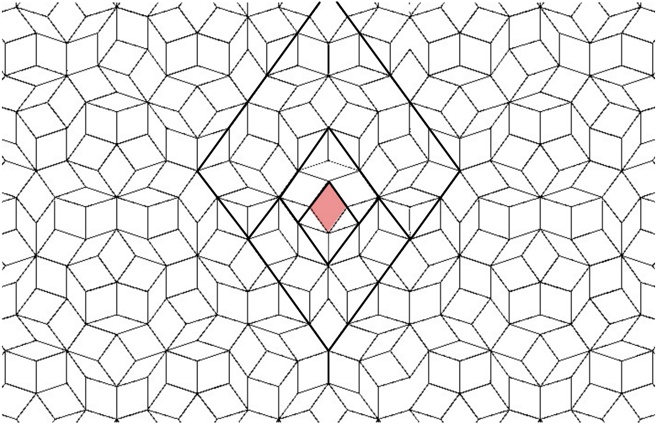
(Remark. The above is not entirely true: this is a simplified discussion.)

How many different Penrose tilings are there?

There are infinitely many different Penrose tilings:

$$\boxed{\begin{array}{c}\{\textbf{DNA Sequences}\}/(\text{identify if eventually agree})\\ \updownarrow \\ \{\textbf{Penrose tilings}\}/(\text{identify if rotate or translate})\end{array}}$$

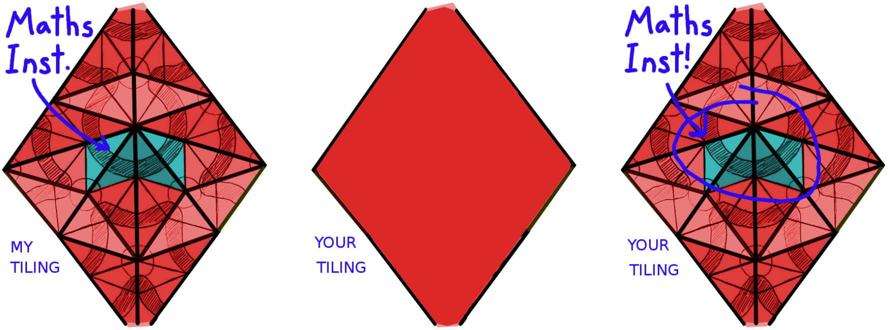
Example: **DDDDD$\cdots$ is the Cartwheel tiling:**



The Maths Institute tiling arises in every Penrose tiling

Any finite patch of a Penrose tiling occurs infinitely many times inside any other Penrose tiling.

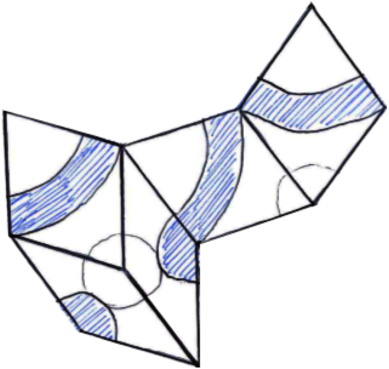
Sketch Proof. Run Steps 1-4 until the region lies inside, say, a huge Dorothy tile. Run same Steps in your tiling, you also have a huge Dorothy. Reverse the Steps to obtain the region (Key Trick).



So you cannot tell two Penrose tilings apart by just looking at a finite patch!

A puzzle for you

Puzzle: **Suppose you've built some finite patch using Penrose tiles.**
Can a computer tell you in finite time whether you'll get stuck?



"This mission is too important for me to allow you to jeopardize it"
(HAL 9000, A Penrose Tiling Odyssey)

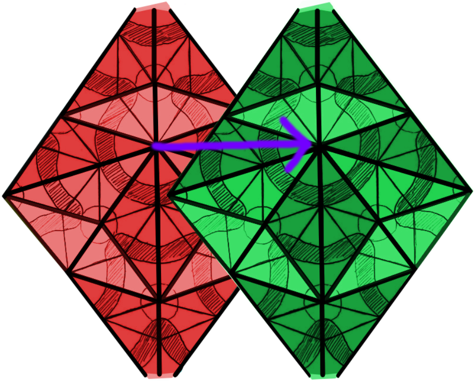
For an answer, see:
people.maths.ox.ac.uk/ritter/masterclasses.html

Thank you for listening

Three additional topics that did not make it into the talk:

1. Why are Penrose rhombi an aperiodic tile-set?
2. What was the Remark "The above is not entirely true" about, in the discussion of reconstructing the tiling from the DNA?
3. How might one have discovered the Penrose tiles?

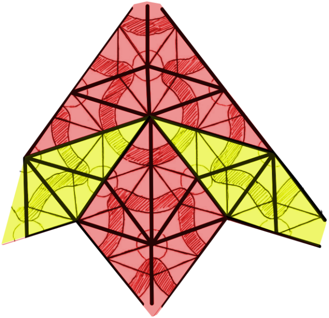
Why are Penrose rhombi an aperiodic tile-set?



Sketch Proof. Suppose the tiling had a translation symmetry. Steps 1-4 and the reversed Steps are **unique**: this implies that the tiling by 60% larger tiles has the same translation symmetry (moving by the **same** distance). Apply Steps 1-4 many times, until you get huge tiles. This tiling has that same translation symmetry. But moving a huge tile by a (by comparison small) distance gives an overlap and tiles are not allowed to overlap!

The Remark “The above is not entirely true”

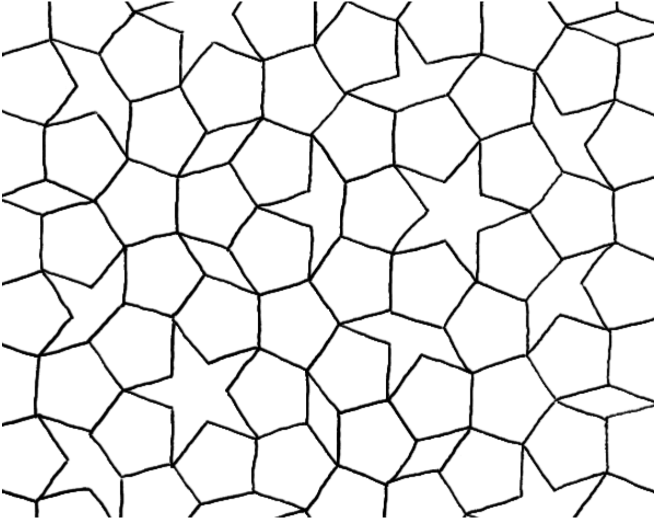
On the slide about DNA reconstruction, it is not true that the letters uniquely tell you which tile to pick. For example, **DDND**:



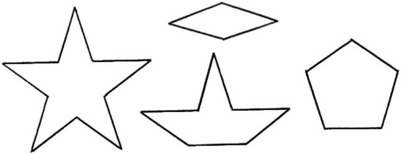
So in reverse, **N** tells you to pick “the” yellow tile: but there are two yellow tiles! To fix this, one distinguishes the triangular pieces obtained by dividing **D,N** tiles in Step 1. Then DNA sequences use letters $\mathbf{D_1}, \mathbf{D_2}, \mathbf{N_1}, \mathbf{N_2}$, corresponding to those triangular tiles. There are rules governing the order in which letters can appear.
(See: **people.maths.ox.ac.uk/ritter/masterclasses.html**)

How the Penrose tiles may have been discovered

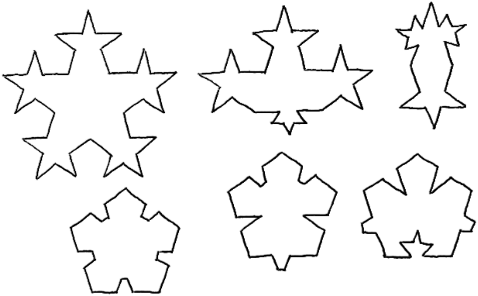
You try to tile the plane by **regular pentagons** (which is impossible), and you fill in the gaps:



Then you need to fine tune the tile set



is not an aperiodic tile-set, but the following is:



(you can then replace indentations by decorations).

Tilings by these give rise to a tiling by Penrose rhombi and vice-versa.
The key is to spot the rhombi in the tiling by pentagons/pentacles.