OXFORD MASTERCLASSES IN GEOMETRY 2014.
Exercises on Geometry, Topology, and Penrose tilings.

Prof. Alexander F. Ritter.

Comments and corrections are welcome: ritter@maths.ox.ac.uk

Spaces and distances.

Exercise 1. *Consider the plane* $\mathbb{R}^2$ *with the distances*

$$d_n(p,q) = \|p-q\|_n = (|p_1-q_1|^n + |p_2-q_2|^n)^{1/n} \qquad (1 \le n < \infty).$$

On one sheet of paper, draw the picture of the unit circle for the distance d_n *as you vary* n*. For example, for* $n=2$ *you get the usual Euclidean distance, and the usual Euclidean circle.*

What happens for large values of n*?*

Show that for any point p*, the collection of neighbourhoods of* p *that* d_n *determines is the same collection for any* n*.*

So you have shown that the topology of $\mathbb{R}^2$ does not depend on the choice of d_n.

Slightly harder: show that you get the same topology on $\mathbb{R}^2$ *(or indeed any* $\mathbb{R}^m$*) for any distance function* d *which is compatible with addition and rescaling, meaning:*

$$d(p,q) = d(p-q,0) \qquad d(kp,0) = k\cdot d(p,0) \text{ for any } k\in\mathbb{R}.$$

In this case, $\|x\| = d(x,0)$ *is called a* norm.

Exercise 2. Recall that on the disc $D = \{z\in\mathbb{C} : |z|<1\}$ we defined a hyperbolic distance d determined by the density function

$$f(z) = \frac{2}{1-|z|^2}.$$

Show that for a Euclidean segment with end-points $z, z+(a+ib)$ *(for small* $a,b\in\mathbb{R}$*), the hyperbolic distance satisfies* $d(z,z+a+ib) \ge d(z,z+a)$ *with equality if and only if* $b=0$*.*

Consider a path from 0 *to a real number* z_0*. By approximating the path by a polygonal path, using small Euclidean segments, show that the hyperbolic length of the path is minimal if the path is a Euclidean-straight line segment along the real-axis from* 0 *to* z_0*.*

So far, we've found the hyperbolic "lines" joining 0 to z_0. Now we want to find the hyperbolic "line" joining any z_1, z_2 by using a symmetry S which maps $S(0)=z_1, S(z_0)=z_2$ (for some z_0).

Convince yourself that

$$S(z) = \frac{az+b}{\bar{b}z+\bar{a}}$$

for $a,b\in\mathbb{C}$ *with* $|a|^2-|b|^2\neq 0$*, maps* D *bijectively onto* D*.*

Show that S *preserves hyperbolic lengths.*[1]

Notice that

$$S(z) = \frac{z+z_1}{\overline{z_1}z+1}$$

is of the above form, and sends $S(0)=z_1$*. Observe that by composing with a further symmetry* $S(z)=e^{i\alpha}z$ *(using* $a=e^{i\alpha/2}, b=0$*), we can assume* $S(z_0)=z_2$ *for a suitable real number* z_0*.*

Date: This version of the exercises was created on September 8, 2014.

[1] *Hint. You need to see how the density function* $f(z)$ *compares with* $f(Sz)$*, but you also need to check how Euclidean distances stretch. That is, for a segment with end-points* $z, z+w$*, you need to compare the Euclidean lengths* $\|z-(z+w)\| = \|w\|$ *and* $\|Sz - S(z+w)\|$*.*

Finally, check that such S will send the real-axis to a Euclidean-circle perpendicular to the boundary circle ∂D (or, in some cases, to a Euclidean-straight line perpendicular to ∂D, but we can think of this as a Euclidean-circle of infinite radius).

Deduce that the hyperbolic "lines" in D are arcs of Euclidean-circles perpendicular to ∂D.

Exercise 3. *Show that the map*

$$z \mapsto -i \cdot \frac{z-i}{z+i}$$

maps the disc $D = \{z \in \mathbb{C} : |z| < 1\}$ bijectively onto the upper half-plane

$$H = \{z : \operatorname{Im} z > 0\},$$

Here $z = x + iy$ corresponds to the coordinates $(x, y) \in \mathbb{R}^2$, and the Imaginary part is $\operatorname{Im} z = y$.

Show that the density function for hyperbolic distances in D becomes the density function

$$f(z) = \frac{1}{y}$$

in H via the above bijection. (Hint. first check how Euclidean distances stretch via the bijection.)

Using your knowledge of hyperbolic "lines" in D, show that the hyperbolic "lines" in H are either half-circles perpendicular to the real axis which bounds H, or half-lines $x =$ constant perpendicular to the real axis.

Tilings.

Exercise 4. *A tiling is called* periodic *if there are at least two translations*[2] *in non-parallel directions that preserve the tiling. Show that for a periodic tiling, there is a rectangular region R of the tiling such that the whole tiling can be recovered from translated copies of R.*

A tiling is called non-periodic *if there is no translation which preserves the tiling. Draw a tiling of the plane which is non-periodic.*

Exercise 5. *Show that, in Euclidean geometry, it is possible to tile the plane using one fixed regular polygon, if and only if the regular polygon is a triangle, a square or a hexagon.*

Show that, in hyperbolic geometry, it is possible to tile the hyperbolic "plane" D using a regular polygon with n sides, for any $n \geq 3$, provided you choose the size of the tile carefully.
(Hint. compare the angles that a hyperbolic regular square has when the square is very small versus when the square is very large.)

Hausdorff distance.

Exercise 6. Recall we defined Hausdorff distance in lectures,[3] that is a notion of distance between two closed[4] bounded subsets S_1, S_2 in $\mathbb{R}^2$.

Given a closed bounded subset $S \subset \mathbb{R}^2$, define its δ-fattening by

$$S^\delta = \{p \in \mathbb{R}^2 : d_{Euclidean}(p, s) \leq \delta \text{ for some } s \in S\},$$

namely those points of $\mathbb{R}^2$ within distance $\delta \geq 0$ from S. Call $D(S_1, S_2) = \min \delta \geq 0$ such that $S_1 \subset S_2^\delta$, the smallest fattening of S_2 necessary for S_1 to be eaten up by S_2^δ. Check that

$$d(S_1, S_2) = \max\{D(S_1, S_2), D(S_2, S_1)\}.$$

Show that the Hausdorff distance satisfies the requirements in the definition of distance functions.

[2] A *translation* of the plane is a map of the form $(x, y) \mapsto (x + a, y + b)$ for some fixed constants a, b. We will always assume that our translations are non-trivial: that is, at least one of a, b is non-zero.

[3] $d(S_1, S_2) = \min \delta \geq 0$ such that any point s_1 of S_1 lies at Euclidean-distance at most δ from some point of S_2, and every point s_2 of S_2 lies at Euclidean-distance at most δ from some point of S_1.

[4] A subset $S \subset \mathbb{R}^2$ is *closed* if its complement $\mathbb{R}^2 \setminus S$ is open. More intuitively: it means that S contains its own boundary points: $\partial S \subset S$.

Consider the Koch snowflake S (see Wikipedia for nice pictures). Recall this is a fractal you obtain by recursively doing a cut and draw operation, so you think of the Koch snowflake as a "limit" of subsets $S_1, S_2, S_3, \ldots$ of $\mathbb{R}^2$ starting from an equilateral triangle S_1.

Using the Hausdorff distance, show that this is mathematically correct: that is, the Hausdorff distances $d(S_n, S) \to 0$ shrink to zero as n grows to infinity. Notice that the Hausdorff distance allows us to measure how "imperfect" the approximation S_n is to the actual Koch snowflake S.

Continuity.

Exercise 7. *Show that a map $f : X \to Y$ between metric spaces is continous if and only if the preimage*[5] *of any open ball around $f(p)$ in Y is a neighbourhood of p in X.*

Exercise 8.

Show that the operation of multiplication $g_\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g_\times(a_1, a_2) = a_1 \cdot a_2$ is continuous.

Show that if $f\colon X \to Y$, $g\colon X \to Z$ are continuous then $h(x) = (f(x), g(x))$ is continuous as a map $h\colon X \to Y \times Z$.

Show that you can view kf, $f+g$, $f-g$, $f \cdot g$ as compositions of continuous functions, and deduce that they are continuous.

Deduce that polynomials are continuous functions.

Exercise 9. *Using the fact that $f : X \to Y$ is continuous if and only if $f(x_n) \to f(p)$ as $x_n \to p$, show that compositions of continuous functions are continuous.*

Exercise 10.

Sketch the graph of the function $f\colon (0,1) \to \mathbb{R}$, $f(x) = \sin(2\pi/x)$ (it may help to consider first the fractions $x = 1/n$ and $x = 2/n$).

Is f continuous?

Suppose we define $f\colon [0,1] \to \mathbb{R}$ by the same formula for $x \neq 0$, and we define $f(0) = 1$.

Is f continuous?

Connected and path-connected spaces.

Exercise 11. *Can you think of a connected space which is not path-connected?*

Maps which contract distances.

Exercise 12. *A map of Oxford is laid out on your work table in Wadham College. Show that there is exactly one point of the table which lies precisely at the correct point represented on the map!*

Continuous deformations. [6]

Exercise 13.

Show that a torus with one point removed can be continuously deformed into a figure eight loop.[7]
Then show that if your bicycle tire has a hole, you can continuously deform the tire inside-out.[8]

Exercise 14.

You have bought two American doughnuts[9] T_1, T_2.
You bite a hole out of your second doughnut T_2 (marked with a cross).
In retaliation, the second doughnut decides to eat the first doughnut.
Show that T_2 can continuously eat T_1 until T_1 lies inside T_2 as shown in the middle picture.

---

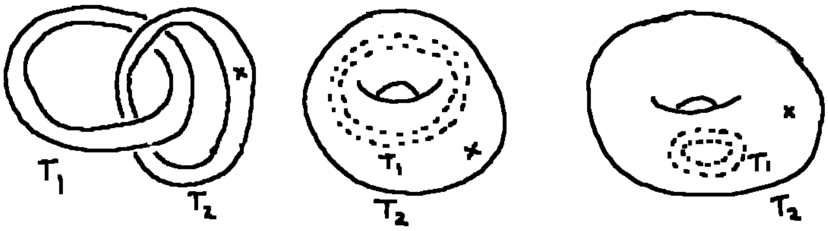
[5] The preimage of a subset $S \subset Y$ means $f^{-1}(S) = \{x \in X : f(x) \in S\}$.

[6] Intuitively, think of playing with objects made out of play-dough. So you can deform dramatically but you are not allowed to puncture holes or tear.

[7] A *figure 8* loop is made up of two circles joined at one point, just like the symbol 8.

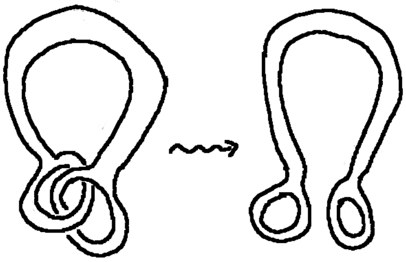
[8] Meaning: the "inner surface" becomes the "outer surface", and vice-versa, after the deformation.

[9] We only consider the surfaces, so the inside is hollow.



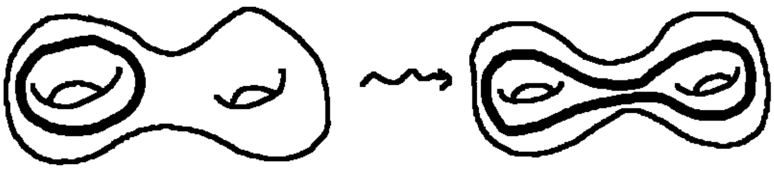
Is it possible for T_2 to eat T_1 in such a way that T_1 will end up inside T_2 as shown in the right-most picture, or would that cause a discountinuous indigestion for T_2?

Exercise 15. *Using a continuous deformation, find a way to undo the following locked handcuffs:*



Exercise 16. *The handcuffs have now been tied to a steering-wheel. Find a way to remove one of the handcuffs from the wheel:*

Exercise 17. *You have bought a doughnut with two holes. But the shop made a mistake: they only sprinkled a loop of sugar around one hole. You are upset. Can you find a way to deform the doughnut, so that your sprinkled sugar loop actually runs around both holes?*



Limits.

Exercise 18. *Consider a closed square S in $\mathbb{C}$. Suppose a grasshopper jumps around on the square. Its positions are $z_1, z_2, z_3, \ldots$.*

Chop up the square into 4 equal squares. Use the pigeonhole principle to show that for one of these four smaller squares, say S', the grasshopper is in S' infinitely many times (at positions, say, $z_{i_1}, z_{i_2}, \ldots$ for some indices $i_1 < i_2 < \cdots$).

Use this chopping idea inductively to show that, some subsequence of positions $z_{j_1}, z_{j_2}, \ldots$ of the grasshopper converges to some point z in the square S. In other words, the grasshopper jumps infinitely many times arbitrarily close to z.

Show in general, that for any closed bounded region $S \subset \mathbb{R}^n$, and any sequence of points $z_1, z_2, \ldots$ in S, there is a subsequence $z_{j_1}, z_{j_2}, \ldots$ which converges to some point z in S (z need not be unique).

Cultural remark. A (metric) space is called *compact* if any sequence of points has a subsequence which converges to a point in the space.

The space of loops and their continuous deformations.

Exercise 19. *Show that any loop $S^1 \to \mathbb{R}^2 \setminus 0$ can be continuously deformed to a loop $S^1 \to S^1$ (viewing the second S^1 as $S^1 \subset \mathbb{C} \setminus 0 = \mathbb{R}^2 \setminus 0$).*

Exercise 20. *Consider the space of loops $f : S^1 \to S^1$ satisfying $f(1) = 1$ (the basepoint).*

Call two loops f, f' equivalent if you can continuously deform one into the other while keeping the basepoint fixed.[10]

Show that you can define a multiplication operation on (equivalence classes of) loops by "concatenating" loops.[11]

Show that the set of equivalence classes of loops under this multiplication forms a group. This group is called the fundamental group *of S^1 and is denoted $\pi_1(S^1)$.*

Using winding numbers, show that you can identify $\pi_1(S^1)$ with the group of integers $\mathbb{Z}$ with the operation of addition.

Exercise 21. *Try to run the idea of the previous exercise for loops $f : S^1 \to X$ (and a choice of basepoint $x \in X$, so $f(1) = x$) for the torus $X = T^2$ (doughnut) and for the figure eight loop $X = \{z \in \mathbb{C} : |z - 1| = 1\} \cup \{z \in \mathbb{C} : |z - (-1)| = 1\}$.*

How do the two groups $\pi_1(X)$ differ?

Hint. It may help to try to build an "exponential function" $E : Y \to X$ (having properties similar to the map $E : \mathbb{R} \to S^1, E(s) = e^{2\pi i s}$ built in the lectures), by trying to draw an appropriate space Y in each case.

The intermediate value theorem.

Exercise 22.

Show that if $f : X \to Y$ is a continuous map between spaces, and $S \subset X$ is a connected subspace, then the image $f(S) \subset Y$ is connected.

Deduce that a continuous function $f : [a, b] \to \mathbb{R}$ will hit[12] *each value between $f(a)$ and $f(b)$.* (This is called the Intermediate Value Theorem.)

Exercise 23.

Given a nice bounded region in the plane, show that one can draw a straight line so that the region in divided up into two smaller regions having equal areas.

Given a loaf of bread, of any shape, show that you can cut it with a straight knife cut so that the loaf gets divided into two pieces of equal weight.

Given two watermelons, of any shape, placed in any way on a table, can you make a straight samurai chop to get two pieces of equal volume?

You now arm yourself with two samurai swords. You practice at making a super-chop*: simultaneously each sword makes a straight cut while keeping the swords perpendicular to each other. As a final performance, show that you can super-chop a cake of any shape so that you get four slices of cake of the same volume.*

Winding numbers.

Exercise 24.

Given a loop $f : S^1 \to \mathbb{C}$, we defined in lectures the winding number $W(f; z)$ of the loop around any point z not on the loop.

Show that $W(f; z)$ does not change if we move z without crossing the loop.[13]

---

[10]So $F : [0, 1] \times S^1 \to S^1$, $F(0, z) = f(z)$, $F(1, z) = f'(z)$, with basepoint $F(t, 1) = 1$ for all times $t \in [0, 1]$.

[11]So $f \# f'$ means: first go around loop f then go around loop f'. Make this precise.

[12]Given any point c between $f(a), f(b) \in \mathbb{R}$, the equation $f(x) = c$ has at least one solution $x \in [a, b]$.

[13]*Hint. Rephrasing the question: show that, for that given f, the winding function w, defined by $w(z) = W(f; z)$, is constant on the connected pieces of $\mathbb{C} \setminus$ (loop).*

Exercise 25.

Find a way to hang[14] *a painting on the wall using two nails, so that removing one nail from the wall will make the painting fall.*

Can you describe in general all ways of hanging up the picture so that this happens?

How would you do it if you had three nails?

Exercise 26.

Suppose you have two loops $f, g : S^1 \to \mathbb{C} \setminus 0$ *such that* $|f(z) - g(z)| < |f(z)|$. *Show that the winding numbers around zero are equal:* $W(f) = W(g)$.

Deduce that the winding number of the Moon around the Sun is the same as the winding number of the Earth around the Sun.

Tilings by Kites and Darts.

Exercise 27. *Show that any parallelogram can tile the plane periodically.*

Show that without markings on the Penrose Kites and Darts, you can periodically tile the plane.

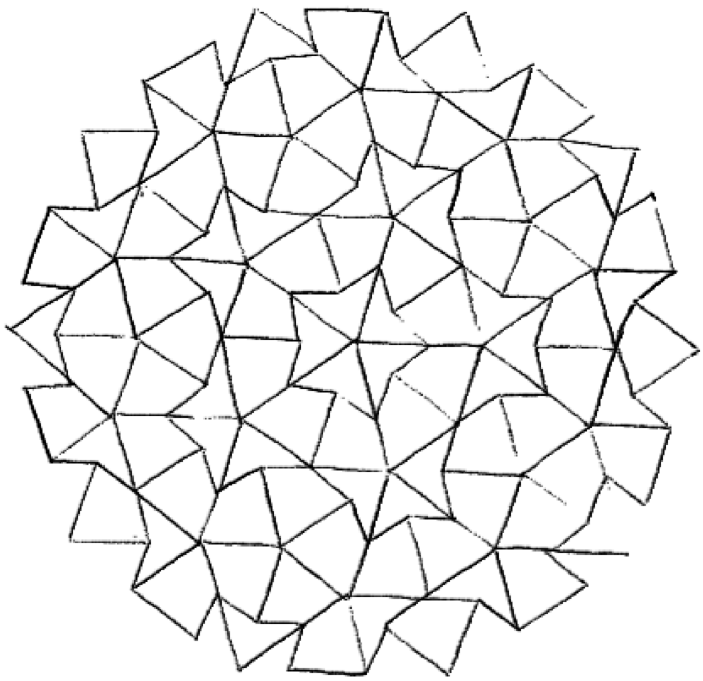
Show that you can make indentations[15] *on the Kite and Dart tiles so that the two deformed tiles will automatically obey the matching rules (without colouring vertices or putting other markings).* So you really do get two honest tiles without markings which are aperiodic.

Composition and Inflation.

Exercise 28. *Apply the inflation procedure from lectures a few times, starting with an* Ace *(the quadrilateral obtained from two Kites locked into a Dart where the* 6α *angle is). You can check your answer, since you should get the* cartwheels $C_1, C_2, \ldots$.

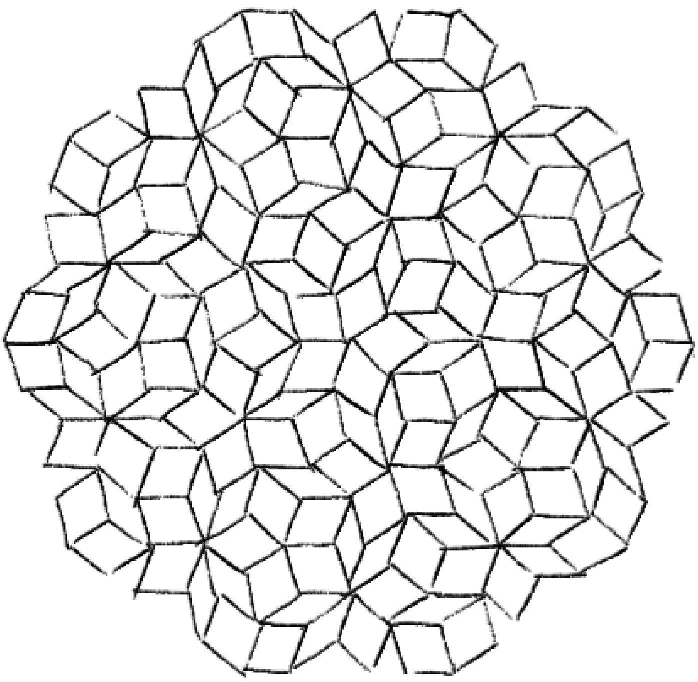
Passing between Kite/Dart tilings and Penrose rhombi tilings.

Exercise 29. *In the following two pictures, the tiling by Kites and Darts is supposed to correspond to the tiling by Penrose rhombi. Check with a pencil that you really can pass from one to the other, in either direction, by following the unique rules mentioned in lectures.*



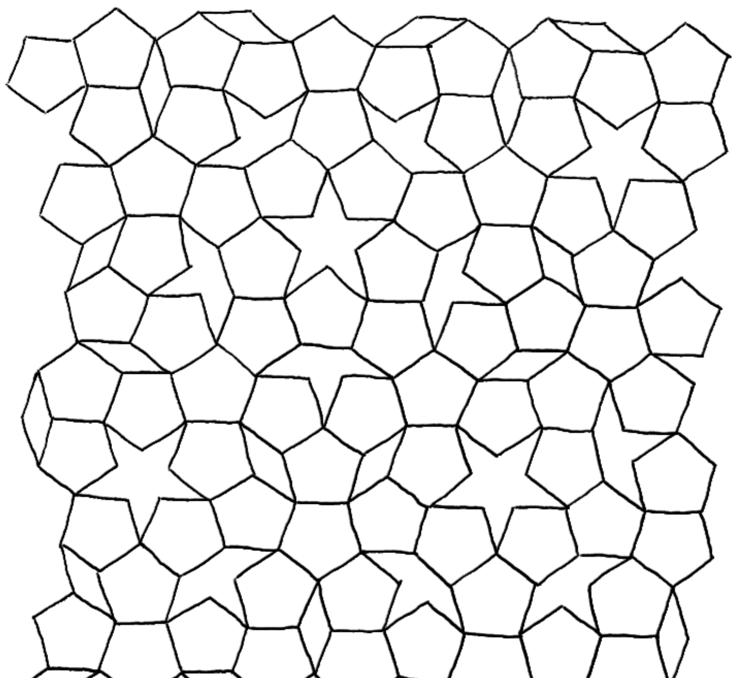
[14]To clarify: you are allowed to wrap the string, which holds up the painting, around the two nails in any complicated way you like. The string is attached onto the painting at the top two corners.

[15]Some small continuous deformation of your two tiles.



Pentagons and pentacles.

Exercise 30 (Harder)**.** *For the following exercise, it is handy to practice with a pencil on this tiling by pentagons/pentacles (the markings are omitted, but you can reconstruct those):*



Using the 6 Penrose pentagons/pentacles tiles, check that the same results we proved in lectures for K, D *still hold for pentagons/pentacles:*

(1) *Find a composition rule and an inflation rule, make sure it is unique;*
(It is easier to work with markings by numbers as explained in the lectures, rather than with the tiles that have spiky decorations on the edges.)
(2) *Show that these 6 Penrose tiles form an aperiodic tile set.*

(3) *Find a way to pass from a tiling by pentagons/pentacles to a tiling by Kites/Darts (or, if you prefer, by Penrose rhombi) and vice-versa.*

Aperiodic tile sets.

Exercise 31 (Open Problem – perhaps impossible)**.**
Is it possible to find just one tile which is an aperiodic tile?

Cartwheels.

Exercise 32. *Using the fact that the cartwheel C_0 sits inside C_2 concentrically, find a nice short clean argument that each cartwheel C_{2n} sits inside C_{2n+2} concentrically.* (Therefore, the cartwheels really do converge to a cartwheel tiling $\lim_{n\to\infty} C_{2n}$)

How many Penrose tilings of the plane are there?

Exercise 33. *Show that there are exactly two non-congruent Penrose tilings which have a (global) five-fold symmetry around a certain centre point.* (Hint. Try building it, step by step.)

Deduce that there are at least two Penrose tilings of the plane which are not congruent.

How do you reconcile the above result, with the fact that any point of any Penrose tiling sits inside arbitrarily large cartwheels C_{2n}? Letting $n \to \infty$, why does that not imply that every Penrose tiling is congruent to the cartwheel tiling $\mathrm{limit}(C_{2n})$*?*

[Very Hard:] *How many non-congruent Penrose tilings of the plane are there?*[16] *Is there a way to classify them?*

Exercise 34 (You never know inside which Penrose tiling you are)**.**

You are walking around a Penrose tiling of the pavement of some fancy building. Show that you can never know which Penrose tiling the architect would have built, if the building had been infinitely large.

Congruent copies of finite regions of Penrose tilings aren't too far away.

Exercise 35. *By considering C_4, show that in any Penrose tiling by Kites and Darts, the maximal distance between two vertices which have the same vertex neighbourhood type is $3 + 2\phi = \phi^4$.*
By carefully looking at the proof that congruent copies of any finite region R of any Penrose tiling arise infinitely often in the tiling, show the following:

Show that given a finite region R, you can find another region R' congruent to R, so that there are some points $p' \in R'$ and $p \in R$ within distance

$$d(p, p') \leq \phi^5 \cdot \mathit{diameter}(R).$$

Deduce that if you rescale the region R by a factor of $2\phi^5 \approx 22.18$, then this rescaled region intersects a different congruent copy of R.

[Harder:[17]] *Penrose and Ammann claim that you can improve ϕ^5 to $\frac{1}{2} + \phi$, which is slightly larger than 2. Can you prove this?*

Density of tiles.

Exercise 36. *Given a periodic tiling using just two tiles, X and Y, show that the ratio of the number of X-tiles over the the number of Y-tiles is a rational number.*

Exercise 37. *Finish the proof from class that the ratio of the number of Kites over Darts in a Penrose tiling converges to the golden ratio ϕ. What you need to check is:*
Show that for larger and larger convex regions, the number of K's over D's approaches ϕ.

Hint. Inside a convex region, find a huge region E which can be exactly tiled using tiles from the tiling T^n (here T^n is the tiling obtained from the original tiling T by doing n inflations).

[16]For example, start with the question: are there only finitely many non-congruent tilings, or infinitely many? If infinitely many, is that countably or uncountably infinitely many?

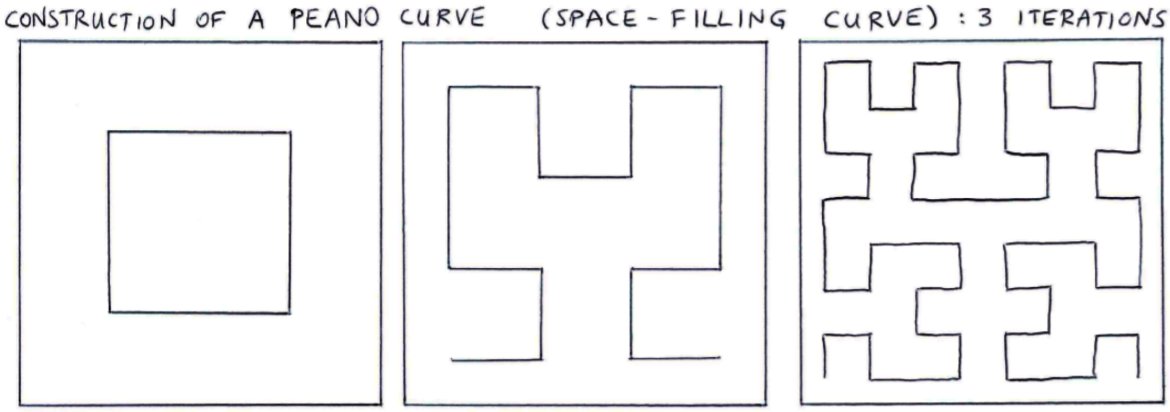
[17]In fact, I haven't checked whether this is true, but it is quoted in some places.

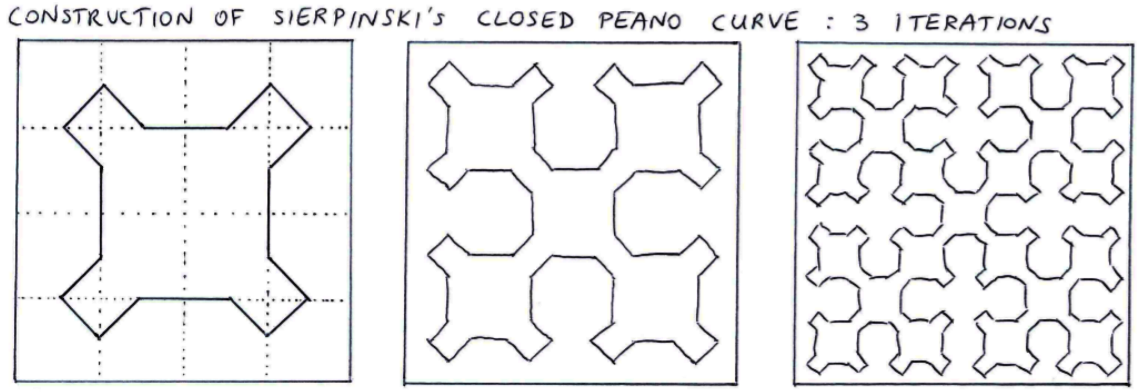
Average numbers of Kites and Darts.

Exercise 38. *Given any tiling by Kites and Darts, and given any $r \in [0, \infty]$ (even 0 or ∞). Is it possible to find larger and larger (non-convex!) regions R_n, made up of tiles, such that they are nested (meaning $R_n \subset R_{n+1}$) and in the limit they cover the plane (meaning $\mathbb{R}^2 = \cup R_n$) but the ratio of the numbers of Kites and Darts in R_n approaches r?*

Space-filling curves.

Exercise 39. *Peano curves are space-filling curves which can be obtained by repeating infinitely many times a simple recipe. For example:*





Show that, as you iterate the construction of the Sierpinski closed Peano curve inside the unit square, the area inside the curve at each stage is getting closer and closer to $5/12$.

Sierpinski's closed Peano curve is in fact a continuous curve which passes[18] *through every point of the square. You will need to accept the fact that it is continuous (which is not so obvious), but you should try checking that indeed the limit curve passes through every point.*

Why does this not contradict the Jordan curve theorem? (Can you prove your answer?)

Build a continuous curve[19] *which passes through each point of a cube (including the inside).*

Build a continuous curve which passes through every point in the whole plane $\mathbb{R}^2$ (or indeed, in $\mathbb{R}^n$ for any given dimension n).

Exercise 40. *Show that there is a continuous loop $f : S^1 \to S^2$ in the sphere which passes through every point of the sphere.*
(Amazing consequence: recall we sketched the proof in class that any loop can be filled continuously with a disc. So even this crazy loop can be filled.)

[18]Given any point p of the square, there is a time $t \in [0, 1]$ such that the curve $c : [0, 1] \to$ Square satisfies $c(t) = p$.

[19]A continuous curve in X means a continuous map $[0, 1] \to X$ (or $[a, b] \to X$ for any chosen $a < b$ in $\mathbb{R}$).