EXAM TOPICS.

PART III, MORSE HOMOLOGY, 2011

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1.1. Important Definitions.

- (local) diffeomorphism, regular map, critical/regular point/value;
- transversality of maps/submfds,
- intersection numbers,
- Morse function, Morse index,
- Baire/generic set,
- Fredholm map, Fredholm index,
- flow/flowline, $-\nabla f$,
- k-handle, handle attachment
- unstable/stable mfds $W^u(p)$, $W^s(q)$,
- Morse-Smale pair (f, g), moduli spaces $W(p, q)$, $\mathcal{M}(p, q)$
- $W^{k,p}$ spaces (4.1 and statement of Thm 4.4: $W^{k,p}$ in terms of weak derivs),
- the Sobolev setup for the proof of the transversality theorem (4.0),
- adjoint, formal adjoint,
- spectral flow,
- topology on $\mathcal{M}(p, q) = W(p, q)/\mathbb{R}$ and notion of lift (5.1),
- convergence to broken trajectories,
- Morse complex, Morse homology.

1.2. Important statements of theorems.

- inverse function theorem, implicit function theorem,
- Sard's theorem,
- Parametric transversality 1 & 2, genericity of transversality,
- basic facts about Fredholm operators,
- properties of Morse functions,
- Morse functions are dense in the space of functions,
- generic height functions are Morse,
- Morse lemma,
- Sard-Smale theorem,
- handle-attaching theorem and CW structure induced by a Morse function,
- transversality theorem (4.0/4.10: for generic g, the section $F = \partial_s - \nabla$ is transverse to zero section of bundle $E \to U \times G$, so $\mathcal{M}(p, q)$ is smooth mfd),
- index of $\partial_s + A_s$ (spectral flow),
- compactness theorem,
- gluing theorem,
- invariance theorem.

1.3. Important calculations.

- computing Morse index,
- computing Morse homology (S^1, deformed S^2, tilted torus, $\mathbb{R}P^2$, $\mathbb{C}P^n$),
- calculating the dimension of $W(p, q)$ assuming Morse-Smale,

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- describing the compactification of $W(p,q)$, $\mathcal{M}(p,q)$ spaces for tilted torus,
- Poincaré duality,
- Künneth theorem,
- Morse inequalities,
- intersection product.

1.4. **Important proofs.**

- regularity is an open condition (so transversality is an open condition),
- transversality results in 1.4,
- parametric transversality 1 & 2,
- generic height functions are Morse (be careful with the coordinates!),
- Sard-Smale theorem,
- basic facts/energy estimates for $-\nabla f$ flowlines,
- topology of sublevel sets does not change if you do not cross a critical value and it changes by handle-attachments if cross a critical value,
- cellular homology = Morse homology for self-indexing Morse function,
- Sobolev/Rellich for $W^{1,2}$ spaces (4.5),
- transversality theorem (4.10-4.12),
- $\partial_s + A_s$ is Fredholm (4.13-4.14),
- index invariance 4.18 (and how to prove Thm 4.15 in the easy case $A_s = A_{\pm\infty}$ constant for s close to $\pm\infty$),
- compactness results (5.2 - 5.4): C^0_{loc}-cgce in $W(p,q)$ implies cgce in $W(p,q)$, reparametrization trick, Thms 5.3, Cor 5.4
- invariance theorem (only what I sketched in lectures),
- proving $\partial^2 = 0$ for the Morse complex,
- $MH_*(f) \cong H_*(M)$ using invariance and self-indexing case (6.1 (3)).

1.5. **Important homework questions.**

- Hwk 1: tilted torus
- Hwk 2, exercise 1: Hessian
- Hwk 4: Morse lemma, $\|x-p\|^2$ Morse for generic p (careful with coords!)
- Hwk 6, exercise 2: generic C^k fns are Morse
- Hwk 8, exercise 1: Morse function with 2 crit pts $\Rightarrow M$ homeo to sphere
- Hwk 13: transversality for smooth g, $W(p,q)$ smooth $\Rightarrow \mathcal{M}(p,q)$ smooth
- Hwk 17: exponential convergence of solutions at the ends
- Hwk 19: $\#\mathcal{M}(p,q) = S^u(p) \cdot S^s(q)$
- Hwk 21: Morse homology of $\mathbb{R}P^2$, $\mathbb{C}P^n$
- Hwk 22: $MC^*(f)$ is iso to dual of MC_*, it takes ∞ time to reach a crit pt.

1.6. **Non-examinable topics.** These topics will definitely **not** appear in the exam:

- 3.12 Modern generalizations (Floer homology, Seiberg-Witten, etc.)
- 2.1 Lemma C^∞ not Banach, 4.3 proof of general Sobolev embedding theorem (but 4.5 is examinable), 4.4 mollification.
- Thm 4.16 & Corollary 4.16 (Lecture 16): $E^u(B)$ is an attractor in the Grassmannian: the proof is non-examinable, but you should know the statement.
- 6.2 Morse homology for mfds with boundary.
- Lectures 22, 23, 24: spectral sequences, Leray-Serre, Morse-Bott.
- Homeworks: everything that is not mentioned in the important homeworks above is non-examinable.