LECTURE 1: ADVERTISEMENT LECTURE.

PART III, MORSE HOMOLOGY, 2011

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What are nice functions?

We will consider the following setup:

$M =$ closed[1] (smooth) m-dimensional manifold,
$f : M \to \mathbb{R}$ a smooth function.

Locally, $f : \mathbb{R}^m \to \mathbb{R}$ near $p = 0$:

$$\begin{aligned} f(x) &= f(0) + \sum_i \frac{\partial f}{\partial x_i}(0) \cdot x_i + \frac{1}{2!} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} + \cdots \\ &= f(0) + df_0 \cdot x + \frac{1}{2} x^T \cdot \mathrm{Hess}_0(f) \cdot x + \cdots \qquad \text{(matrix notation)} \end{aligned}$$

Hope:

- Want few $p \in M$ with

  $df_p = 0 \qquad \longleftarrow p$ **critical point** (e.g. max,min)

  Note: these are m conditions, so we hope

  (i) finite $\mathrm{Crit}(f) = \{\text{critical points of } f\}$

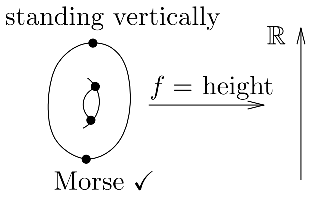
- At critical p, we want a good next order term:

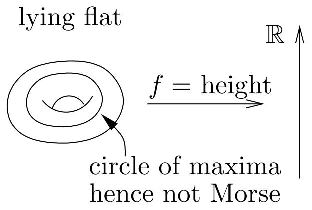
  (ii) $\det \mathrm{Hess}_p(f) \neq 0 \quad \longleftarrow p$ **nondegenerate**

Fact. *(ii)* $\Rightarrow$ *(i)*

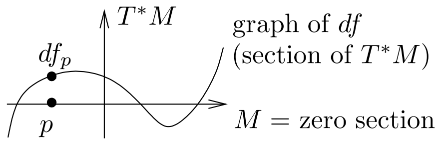
Def. $\boxed{f : M \to \mathbb{R} \textit{ is Morse if all critical points are nondegenerate}}$

Example. $M =$ *torus:*





Modern perspective



$\boxed{f \text{ is Morse} \Leftrightarrow \begin{array}{l}\text{the section } df \text{ is transverse}\\ \text{to the zero section of } T^*M\end{array}}$

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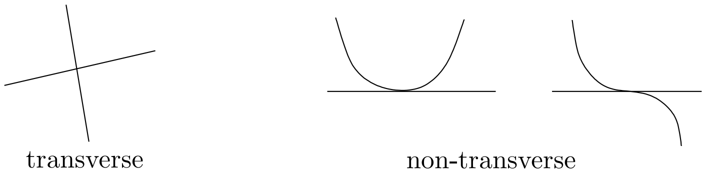
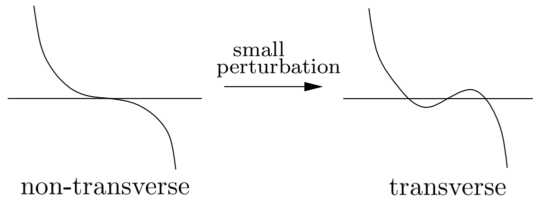
[1]closed = compact and no boundary.

Idea of what "transverse" means:



Transverse objects are the "generic" ones in geometry:



$$\Rightarrow \boxed{\text{almost all functions are Morse}}$$

Fact. *All manifolds arise as submanifolds of some $\mathbb{R}^k$. We'll prove that: almost any "height function"*[2] *is Morse on $M \subset \mathbb{R}^k$.*

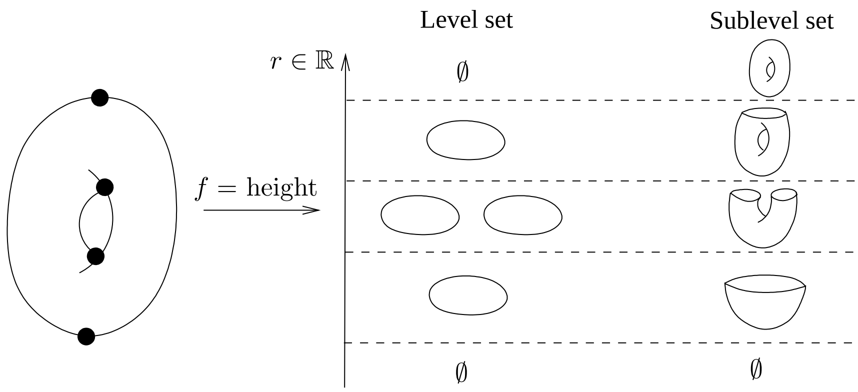
$$\Rightarrow \boxed{\text{it is easy to find Morse functions}}$$

How do you relate Morse f to the topology?

Two natural geometrical objects to look at, given a function f:

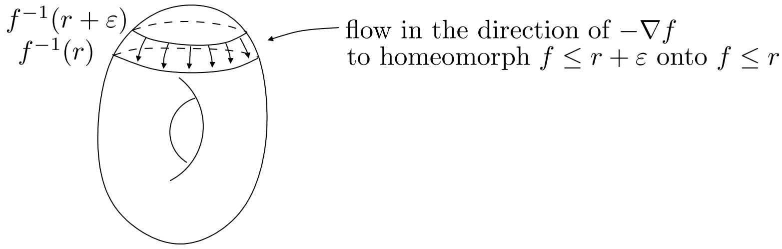
level sets	$f = r$
sublevel sets	$f \leq r$

Consider the torus standing vertically:



Observe that the topology changes when you cross the **critical values** $f(p)$, where $df_p = 0$. Otherwise the topology does not change:

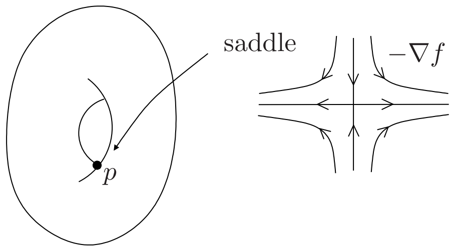
[2]height function = linear functional $\mathbb{R}^k \to \mathbb{R}$.



We will prove: passing through a critical point changes the topology by attaching a k-cell, where

$$k = \#\text{negative eigenvalues of } \mathrm{Hess}_p f \quad \longleftarrow \textbf{index } |p|$$

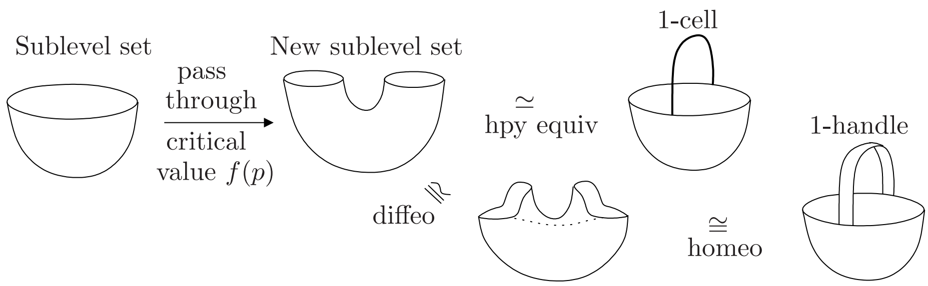
Example: M = torus, f = height



locally $f(x,y) = -x^2 + y^2$

$$\mathrm{Hess}_0 f = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

so $|p| = 1$



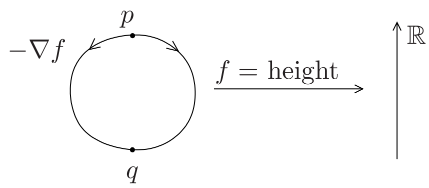
$\Rightarrow$ Can reconstruct the mfd up to hpy equivalence

Hwk 8: f Morse with 2 critical points $\Rightarrow M \cong S^m$ homeomorphic.
Warning. Not diffeomorphic, $\exists$ "exotic" S^7 homeo but not diffeo to the usual $S^7 \subset \mathbb{R}^8$ (proved by Milnor, using the result of Hwk 8).

Is it easy to recover the homology from f?

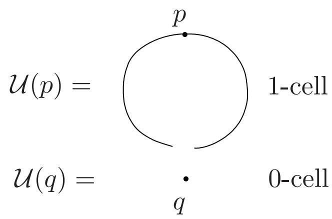
Classical approach (Morse $\sim$ 1930, Thom, Smale, Milnor $\sim$ 1960, ...)
Pick a self-indexing Morse function (meaning index$(p) = f(p)$).
$\Rightarrow$ the above cell-attachments define a CW structure on M.
$\Rightarrow$ recover cellular homology of M

Example: $M = S^1$



Consider the unstable cells

$$x \mapsto \mathcal{U}(x) = \{\text{points flowing down from the critical point } x\}$$



The cellualar boundary is:[3]

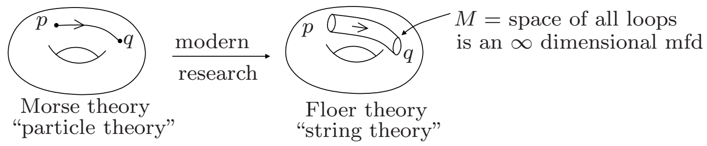
$$\partial\mathcal{U}(p) = \mathcal{U}(q) - \mathcal{U}(q) = 0$$

so $H_*^{\text{cell}}(S^1)$ is generated by cells $\mathcal{U}(q)$,$\mathcal{U}(p)$ in degrees $*$ =0,1, as expected.

Why is the classical approach bad?

If M is ∞-dimensional, then $\mathcal{U}(p)$ is usually ∞-dimensional, hence not a cell. Also, the "flow" is often not defined, so $\mathcal{U}(p)$ is not even well-defined.

You may ask: who cares about ∞-dimensional manifolds? Actually, these nowadays arise quite naturally in geometry. For example:

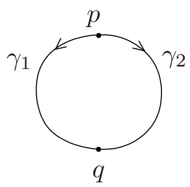


Modern approach (Witten, Floer, . . . ~ 1980)

Consider the **moduli space**[4]

$$\mathcal{M}(p, q) = \{-\nabla f \text{ flowlines from } p \text{ to } q\}/\text{reparametrization}$$

Example: $M = S^1$



$$\mathcal{M}(p, q) = \{\gamma_1, \gamma_2\}$$

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[3]Note that orientation signs are a subtle issue: if we got the sign wrong, then suddenly $\partial\mathcal{U}(p)$ would no longer be zero. To avoid such technical subtleties, we will work over $\mathbb{Z}/2$ in this course.

[4]these flowlines are called *instantons* or *tunneling paths* in physics.

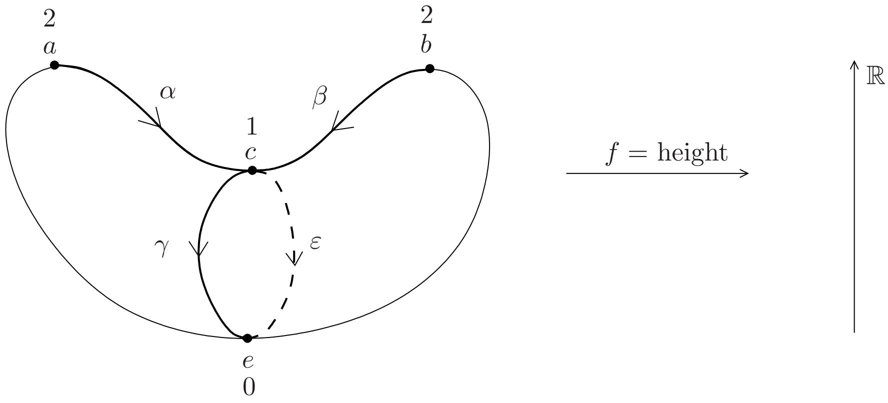
Define a chain complex, called **Morse complex**,

$$MC_*(f) = \mathbb{Z}_2 \cdot p \oplus \mathbb{Z}_2 \cdot q \oplus \cdots$$

where $p, q, \ldots$ are the critical points, and we work over $\mathbb{Z}_2 = \mathbb{Z}/2$ to avoid signs.

$$\begin{array}{l} d : MC_k \to MC_{k-1} \\ dp = \sum_q (\#\text{elements in } \mathcal{M}(p,q)) \cdot q \end{array}$$

Example: M = hot-dog ($\cong S^2$)



Then:

$$\begin{array}{lcl l} MC_2 & = & \mathbb{Z}_2 \cdot a \oplus \mathbb{Z}_2 \cdot b & da = \#\{\alpha\} \cdot c = c,\ db = \#\{\beta\} \cdot c = c \\ MC_1 & = & \mathbb{Z}_2 \cdot c & dc = \#\{\gamma, \varepsilon\} = 0 \pmod 2 \\ MC_0 & = & \mathbb{Z}_2 \cdot e & de = 0 \end{array}$$

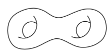
$$\begin{array}{lcccccc} \textbf{Morse homology} = \dfrac{\ker \partial}{\operatorname{im} \partial} & = & MH_*(f) & = & \mathbb{Z}_2 \cdot e & \oplus & \mathbb{Z}_2(a-b) \\ & & * & = & 0 & & 1 \end{array}$$

Observe this is the same as $H_*(S^2)$ (over $\mathbb{Z}/2$).

$$\boxed{\textbf{Theorem. } MH_*(f) \cong H_*(M)}$$

Cor.

$$\begin{array}{lcl} \#(\textit{critical points of a Morse function}) & = & \#(\textit{generators of } MC_*(f)) \\ & \geq & \#(\textit{generators of } MH_*(f)) \\ & = & \sum \dim H_i(M) \quad (\textit{Betti numbers}) \end{array}$$

Example. A generic $f :$  $\to \mathbb{R}$ has $\geq 2 + 2 \cdot \text{genus} = 6$ critical points.[5]

Geometry is functional analysis

We made two tacit assumptions when defining MC_*, MH_*:

(1) need $\#\mathcal{M}(p,q)$ finite for $|q| = |p| - 1$. Rephrasing:

$$\boxed{\mathcal{M}(p,q) \text{ is a compact 0-dimensional manifold}}$$

(2) need $d^2 = d \circ d = 0$ to define homology.

[5] *Non-examinable:* Algebraic topology tells you $\geq |\chi(M)| = |\sum(-1)^i \dim MC_i(f)|$, via the intersection number: $\text{graph}(df) \cdot 0_{T^*M} = -\chi(M)$. So for a torus it just predicts ≥ 0.

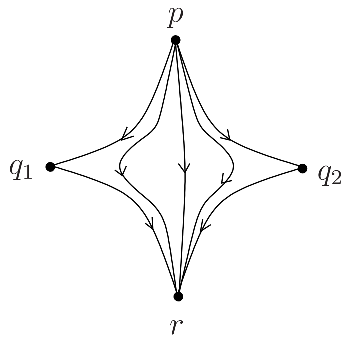
Idea of proof of (2):

$$\begin{aligned} d^2 p &= d(\textstyle\sum_q \#\mathcal{M}(p,q)\cdot q) \\ &= \textstyle\sum_{q,r} \#\mathcal{M}(p,q)\cdot \#\mathcal{M}(q,r)\cdot r \end{aligned}$$

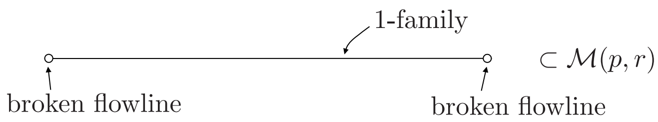
Now $\#\mathcal{M}(p,q)\cdot\#\mathcal{M}(q,r)$ counts "broken" flowlines from p to q to r. Hence:

$$d^2 = 0 \Leftrightarrow \text{once-broken flowlines arise in pairs}$$

Hope: $\exists$ a 1-family of flowlines joining two broken flowlines:



View the flowlines as points in the moduli space, then:



Hope:

- $\mathcal{M}(p,r)$ is a non-compact 1-mfd
- $\exists$ natural way of making it compact:

$$\overline{\mathcal{M}}(p,r) = \mathcal{M}(p,r) \cup \text{“}\partial\mathcal{M}(p,r)\text{”} \quad (\partial\mathcal{M}(p,r) = \{\text{broken flowlines}\})$$

$\Rightarrow \overline{\mathcal{M}}(p,r)$ compact 1-mfd
$\Rightarrow \overline{\mathcal{M}}(p,r) =$ disjoint union of circles and compact intervals
$\Rightarrow \partial\overline{\mathcal{M}}(p,r) =$ even number of points, so $= 0$ mod 2
$\Rightarrow d^2 p = 0$
$\Rightarrow d^2 = 0 \quad \square$

Idea of proof of (1): Functional Analysis

(3) **transversality problem:** $\mathcal{M}(p,q)$ are smooth manifolds for a "generic" metric g (which defines ∇f by $g(\nabla f, \cdot) = df$), and

$$\dim \mathcal{M}(p,q) = |p| - |q| - 1.$$

(4) **compactness problem:** $\mathcal{M}(p,q)$ can be compactified by broken flowlines.

Most modern homology theories involve these two problems

$$\Rightarrow \boxed{\text{Morse homology is a perfect playground!}}$$

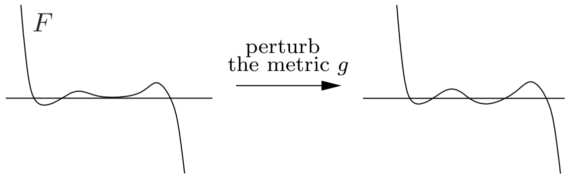
Idea to solve (3): consider the "Banach" vector bundle

$$\begin{array}{c}\{\text{smooth vector fields along } u\} \\ \downarrow \quad \curvearrowright \text{section } F = \partial_s u - \nabla f(u) \\ \{\text{smooth paths } u : \mathbb{R} \to M, u(s) \to p, q \text{ as } s \to -\infty, +\infty\}\end{array}$$

Observe:

$$\boxed{F = 0 \Leftrightarrow \partial_s u = -\nabla f \Leftrightarrow u \in \mathcal{M}(p,q)}$$

$\Rightarrow$ $\boxed{\mathcal{M}(p,q) = \text{intersection of a section of a Banach vector bundle with 0 section}}$



non-transverse

transverse
$\Rightarrow \mathcal{M}(p,q)$ mfd!

Geometry is algebra

Define the Morse cohomology $MH^*(f) \cong H^*(M)$ using

$$\begin{aligned}&MC^* = \mathbb{Z}_2 \cdot p \oplus \mathbb{Z}_2 \cdot q \oplus \cdots \\ &\delta : MC^k \to MC^{k+1} \\ &\delta p = \textstyle\sum_q \#\mathcal{M}(q,p) \cdot q \qquad \text{(where } |q| = |p| + 1)\end{aligned}$$

Then *Poincaré duality* $H_*(M) \cong H^{m-*}(M)$ (over $\mathbb{Z}_2$)[6] is just the symmetry:

$$\begin{array}{rcl} MH_*(f) & \cong & MH^{m-*}(-f) \\ p & \mapsto & p \\ \mathcal{M}(p,q;f) & \cong & \mathcal{M}(q,p;-f) \\ -\nabla f \text{ flowline } u(s) & \mapsto & \nabla f \text{ flowline } u(-s) \end{array}$$

The switch in grading is because flipping the sign of f flips the sign of the Hessian.

Poincaré duality is just reversal of flowlines in Morse theory! If you use a height function, Poincaré duality is the intuitive idea "look at the manifold upside down!".

The *Künneth isomorphism* $H_*(M_1 \times M_2) \cong H_*(M_1) \otimes H_*(M_2)$ can be proved quite simply now by the observation: Morse functions $f_1 : M_1 \to \mathbb{R}$, $f_2 : M_2 \to \mathbb{R}$ give naturally rise to the Morse function $f_1 + f_2 : M_1 \times M_2 \to \mathbb{R}$, and the flowlines are just the combined flowline for f_1, f_2 on the respective factors of $M_1 \times M_2$.

The *cup product* $H^a(M) \otimes H^b(M) \to H^{a+b}(M)$ can also be described Morse theoretically: you count flows along a Y-shaped Feynman graph, flowing by a Morse function along each of the three edges of the graph and you require that the flow converges to the inputs p, q at the top, and to r at the bottom. This solution then contributes $p \cdot q = r + \cdots$ to the product.

[6] this also works over $\mathbb{Z}$, but then one needs to assume M is orientable, which is secretly hidden in the orientation signs that define MC_*, MC^*.