LECTURE 2.

PART III, MORSE HOMOLOGY, 2011

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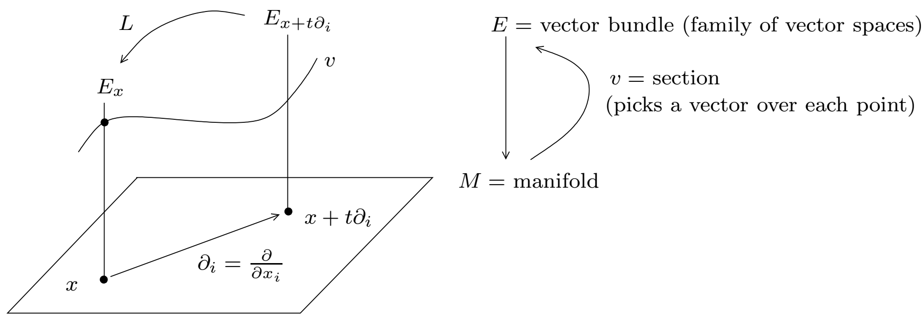
0. REVIEW OF DIFFERENTIAL GEOMETRY

0.1. Connections: motivation. How do you differentiate on vector bundles?



In very loose notation, we want

$$\nabla_{\partial_i} v = \text{“} \lim_{t \to 0} \frac{[L \cdot v(x + t\partial_i)] - v(x)}{t} \text{”}$$

where we need a linear identification $L : E_{x+t\partial_i} \to E_x$. Expanding

$$L = I + tA + \text{order } t^2,$$

since at $t = 0$ we definitely want $L =$ identity. Expanding v:

$$v(x + t\partial_i) = v(x) + t\partial_i v + \text{order } t^2.$$

$$\begin{aligned} \text{So we want} \quad \nabla_{\partial_i} v &= \lim_{t \to 0} \frac{\cancel{v(x)} + t\partial_i v + tAv(x) - \cancel{v(x)} + \text{order } t^2}{t} \\ &= \partial_i v + Av \end{aligned}$$

0.2. Connections.

Notation.[1] $C^\infty(M)$ = smooth functions $M \to \mathbb{R}$
$C^\infty(E)$ = smooth sections $M \to E$ of the vector bundle $E \to M$.

Def. *A* connection ∇ *is a bilinear map*

$$\begin{aligned} \nabla : \quad & C^\infty(TM) \otimes C^\infty(E) \to C^\infty(E) \\ & (X, v) \mapsto \nabla_X v \end{aligned}$$

(1) $C^\infty(M)$*-linear in* X*:* $\nabla_{fX} v = f \nabla_X v$
(2) $C^\infty(M)$*-Leibniz in* v*:* $\nabla_X fv = (X \cdot f)v + f\nabla_X v$

Date: May 1, 2011,

[1] The space of sections is also written $\Gamma(E)$, but we will need to keep track of the topology.

Rmk. *X plays the same role as ∂_i, and $X \cdot f$ is differentiation of f just like $\partial_i f$ would be. Check that $X \cdot f = df(X)$, where the 1-form df "eats" the vector X.*

Equivalent Definition[2] (by defining $\nabla v \cdot X = \nabla_X v$),

$$\nabla : C^\infty(E) \to C^\infty(T^*M) \otimes C^\infty(E)$$

such that it is $C^\infty(M)$-Leibniz: $\nabla(fv) = df \otimes v + f\nabla v$.

Locally

$C^\infty(E)$ has basis $s_1, \dots, s_r$ $\qquad (E|_U \cong U \times \mathbb{R}^r)$

$v = \sum v^j s_j$ $\qquad (v^j \in C^\infty(M))$

$\nabla v = \sum dv^j \otimes s_j + v^j \nabla s_j$

$$\nabla \begin{pmatrix} v^1 \\ \vdots \\ v^r \end{pmatrix} = d \begin{pmatrix} v^1 \\ \vdots \\ v^r \end{pmatrix} + A \begin{pmatrix} v^1 \\ \vdots \\ v^r \end{pmatrix} \qquad (A = \text{“matrix of 1-forms” with } \nabla s_j = \textstyle\sum_k A_j^k s_k)$$

$\boxed{\nabla = d + A}$ locally, as expected!

0.3. **Pull-back bundle.**

$$\begin{tikzcd} u^*E \arrow[r] \arrow[d] & E \arrow[d] \\ {[0,1]} \arrow[r, "u"] & M \end{tikzcd}$$

For $u : [0,1] \to M$ smooth, define $(u^*E)_s = E_{u(s)}$. We will define $u^*\nabla : C^\infty(T[0,1]) \otimes C^\infty(u^*E) \to C^\infty(u^*E)$.
Let s be the coordinate on $[0,1]$, v an E-field along u. We abbreviate $(u^*\nabla)(\partial_s \otimes v)$ by $\nabla_s v = \frac{\nabla}{\partial s} v$

Locally $\boxed{\nabla_s v = \frac{\partial v}{\partial s} + A(\partial_s u) \cdot v}$ (1-forms in A eat $\partial_s u$, $\cdot$ is matrix multiplication)

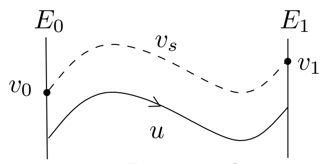
Rmk. *Some abusively write $\nabla_{\partial_s u} v$ because if u is an embedding,*[3] *then any $\widetilde{v} \in C^\infty(E)$ with $\widetilde{v}(u(s)) = v(s)$ satisfies*[4]

$$(\nabla_s v)_s = (\nabla_{\partial_s u} \widetilde{v})_{u(s)} = d\widetilde{v}(\partial_s u) + A(\partial_s u) \cdot \widetilde{v}$$

0.4. **Parallel transport along a path.** For smooth $u : [0,1] \to M$ define

$$\boxed{P_u : E_{u(0)} \to E_{u(1)}}$$

For $v_0 \in E_{u(0)}$ solve the linear ODE $\begin{cases} \nabla_s v = 0 \\ v(0) = v_0 \end{cases}$
in the unknown $v \in C^\infty(u^*E)$. Then $P_u(v_0) = v(1)$.



- ODE theory $\Rightarrow$ $\exists$ unique solution, depending smoothly on u $\Rightarrow$ P_u smooth.
- Linear ODE $\Rightarrow$ P_u linear.

$\Rightarrow$**Hwk 2.** can make Motivation 0.1 rigorous: for a path u with $u(0) = x, u'(0) = X$,

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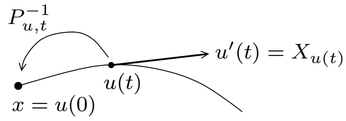
[2] $C^\infty(T^*M) = \Omega^1(M) =$ 1-forms on M, and $C^\infty(T^*M) \otimes C^\infty(E) = \Omega^1(M,E) =$ E-valued 1-forms on M.

[3] $f : M \to N$ is an *embedding* if f is an immersion with $f : M \to f(M)$ a homeomorphism. *Immersion* = the differential df is injective at each point. *Homeomorphism* = continuous bijection having a *continuous* inverse. Often require *proper* = preimages of compact sets are compact (if M, N are compact manifolds this is superfluous, and in this case embedding = injective immersion).

[4] $d\widetilde{v}(\partial_s u) = \frac{\partial}{\partial s}\widetilde{v}(u(s)) = \frac{\partial}{\partial s} v(s)$: so you only need to know how $\widetilde{v}$ varies along $u(s)$.

$$\begin{aligned}(\nabla_X v)_x &= \lim_{t\to 0} \frac{P_{u,t}^{-1}\cdot v(u(t)) - v(x)}{t}\\ &= \frac{\partial}{\partial t}\Big|_{t=0} P_{u,t}^{-1}\cdot v \quad \in E_x\end{aligned}$$



0.5. **Levi-Civita connection.**

$E = TM$, g Riemannian metric[5] $\Rightarrow \exists$ unique ∇ such that ∇ is

(1) symmetric:[6] $\nabla_X Y - \nabla_Y X = [X, Y]$
(2) g-compatible: $X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

Remarks.

Locally: $s_j = \partial_j = \frac{\partial}{\partial x_j}$, $\quad \nabla \partial_j = \sum A_j^k \otimes \partial_k$, $\quad \nabla_{\partial_i}\partial_j = \sum \Gamma_{ij}^k \partial_k$

(Γ_{ij}^k = Christoffel symbols, locally $C^\infty(M) : A_j^k = \sum \Gamma_{ij}^k dx^i$)

(1) $\Leftrightarrow$ locally $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i \Leftrightarrow \Gamma_{ij}^k = \Gamma_{ji}^k$.

(1) + (2) $\Leftrightarrow$ an equation[7] for Γ_{ij}^k in terms of $g^{-1}, \partial g$ holds.

Lemma. *P_u is an isometry for Levi-Civita ∇*

Proof. $\partial_s |v|^2 = \partial_s g(v, v) = g(\nabla_s v, v) + g(v, \nabla_s v) = 0$ (since $\nabla_s v = 0$). ☐

0.6. **Geodesics.**

Consider $E = TM$, ∇ = Levi-Civita.

Def. $u : [0,1] \to M$ is a *geodesic* $\quad \Leftrightarrow \quad \partial_s u$ is parallel along u
$\Leftrightarrow \quad \nabla_{\partial_s u}\partial_s u = 0$
$\Leftrightarrow \quad$ (nonlinear) 2^{nd} order ODE[8] in u.

Cor. *Geodesics have constant speed $|\partial_s u|$. Length$(u) = \int_0^1 |\partial_s u|\, ds = |\partial_s u|$.*

Proof. P_u is an isometry. ☐

The space of sections is also written $\Gamma(E)$, but we will need to keep track of the topology. **Fact.** Geodesics minimize length *locally.*[9] So they can be used to measure distances between closeby points.

0.7. **Exponential map.**

ODE theory $\Rightarrow$ geodesics exist for small time
ODE theory $\Rightarrow$ geodesic is uniquely determined by initial conditions
$\Rightarrow$ call $\exp_p(s, v) = u(s)$ geod with $u(0) = p \in M$, $u'(0) = v \in T_pM$

Easy exercise. $\exp_p(st, v) = \exp_p(s, tv)$
$\Rightarrow$ can write $\exp_p(s \cdot v)$.

Denote $D_\varepsilon TM = \{v \in TM : |v| \le \varepsilon\}$, called a *disc bundle.*

Cor. *For a closed manifold M,*

$$\exp : D_\varepsilon TM \xrightarrow{smooth} M,\ (p, v) \mapsto \exp_p v$$

[5] $g \in C^\infty(\mathrm{Sym}^2(\mathrm{T^*M}))$ fibrewise positive definite (= g is an inner product in each fibre, varying smoothly over M). $\mathrm{Sym}^2(T^*M)$ = bundle of bilinear forms on TM. The norm is $|v| = \sqrt{g(v, v)}$.

[6] locally $X = \sum X^i \partial_i, Y = \sum Y^i \partial_i$ (X^i, Y^i functions in x), $[X, Y] = \sum (X^i \partial_i Y^j - Y^i \partial_i X^j)\, \partial_j$.

[7] Non-examinable: $\Gamma_{ij}^\ell = \frac{1}{2}\sum g^{\ell k}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij})$, where $g^{\ell k}$ = matrix entries of g^{-1}.

[8] Non-examinable: $\partial_s^2 u^k + \sum \Gamma_{ij}^k \cdot \partial_s u^i \cdot \partial_s u^j = 0$.

[9] great circles are geodesics on the sphere, but they don't minimize length after half a circle.

is defined[10] *and distance preserving*[11] *for small* $\varepsilon > 0$.

Proof. Take $\varepsilon =$ smallest t such that $\exp_p(t, v)$ is defined for all $|v| = 1$. Smoothness follows from ODE theory: smooth dependence of solutions on initial conditions. □

Cor. *For M compact, $\exists \varepsilon > 0$ such that any two points in M at distance $< \varepsilon$ are connected by a unique*[12] *geodesic. So any two sufficiently close continuous paths are homotopic by following geodesic arcs.*

0.8. **Differential.** For a (smooth) map of mfds $f : M^m \to N^n$, the derivative map

$$df(p) = d_p f : T_pM \to T_{f(p)}N$$

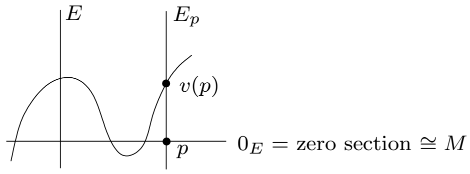
is the best linear approximation of f. Locally $f : \mathbb{R}^m \to \mathbb{R}^n$, $df = \left(\frac{\partial f_i}{\partial x_j}\right)$ (matrix).

Trick. Any vector v at p gives rise to a smooth curve c defined near $c(0) = p$, with $c'(0) = v$ (and vice-versa any c defines a vector $c'(0) = v$ at p). Then

$$\boxed{d_p f \cdot v = \frac{\partial}{\partial s}\bigg|_{s=0} f(c(s))}$$

0.9. **Vertical differential.**

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ M \arrow[u, bend left, "\text{section } v\ (\pi v = \mathrm{id})"] \end{tikzcd}$$



A section $v : M \to E$ has derivative $dv : TM \to TE$. This is ugly because TE has double the dimension of E!

Key Idea:

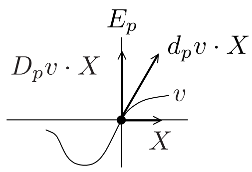
$$T_e^{\text{vertical}}E = \ker d\pi_e \cong E$$

since it is a vector space (analogue of $T_x\mathbb{R}^m \cong \mathbb{R}^m$).

Def. *For ∇ on E, there exists*[13] *a* vertical differential (*or* linearization)*,*

$$\begin{aligned} &D_p : T_pM \to E_p \\ &D_p v \cdot X = (\nabla_X v)_p \end{aligned}$$

Lemma. *$D_p v$ is independent of ∇ if $v(p) = 0$, indeed it is the vertical projection composed with $d_p v$:*



$$\boxed{\begin{aligned} D_p v \cdot X &= d_p v^{loc} \cdot X \\ &= d_p v \cdot X - X \\ &= (\textit{projection to } E_p) \circ d_p v \cdot X \end{aligned}}$$

[10] $\exp : TM \to M$ is *defined* on all of TM (for closed M): patch local solutions of the ODE. But not necessarily on non-closed M: consider $\mathbb{C}\setminus$point and any straight line intersecting 0.

[11] meaning $\mathrm{dist}(p, \exp_p v) = |v|$, using the above Fact.

[12] uniqueness will be explained in Example 1.1.

[13] One could write $D_p = \nabla$. The emphasis is that the definition depends on the *vector* X at p, rather than a *vector field* X defined near p. Also, D_p reminds you about the Lemma.

Proof/Explanation.
Locally: $E|_U \cong U \times E_p$, $\quad T_{(p,v)}E \cong T_pU \times T_vE_p \equiv T_pU \times E_p$, write

$$\boxed{v(x) = (x, v^{\mathrm{loc}}(x)) \in U \times E_p}$$

$$\Rightarrow d_pv \cdot X = (X, d_pv^{\mathrm{loc}} \cdot X)$$

$$\Rightarrow (\nabla_X v)_p = d_pv^{\mathrm{loc}} \cdot X + A(X) \cdot v^{\mathrm{loc}}(p)$$

$$= d_pv^{\mathrm{loc}} \cdot X \qquad\qquad (\text{since } v^{\mathrm{loc}}(p) = 0)$$

In particular,

$$T_pM \equiv T_p0_E \subset TE,$$

so $d_pv \cdot X - X$ makes sense and $\equiv d_pv^{\mathrm{loc}} \cdot X$.

Also

$$T_{(p,0)}E \equiv T_pM \oplus T^{\mathrm{vertical}}_{(p,0)}E \equiv T_pM \oplus E_p$$

independently of coordinates, so vertical projection is defined.[14] □

1. DIFFERENTIAL TOPOLOGY

1.0. **Motivation.** The aim is to relate the infinitesimal, the local and the global:

$$\begin{array}{ccccc}
\textbf{Infinitesimal} & \longrightarrow & \textbf{Local} & \longrightarrow & \textbf{Global} \\
 & \begin{array}{l}\text{linear algebra}\\ \text{inverse function thm}\\ \text{implicit function thm}\\ \text{......}\end{array} & & \begin{array}{l}\text{charts}\\ \text{tubular neighbourhoods}\\ \text{transition rules}\\ \text{partitions of unity}\\ \text{.......}\end{array} &
\end{array}$$

Example:

$$\begin{array}{ccccc}
T_pM & \longrightarrow & \text{Chart} & \longrightarrow & \text{Manifold } M \\
\begin{array}{l} d_pf : T_pM \to T_pM \\ \text{linear isomorphism}\end{array} & & f \text{ local diffeo} & & \begin{array}{l}\text{if } f \text{ also bijective,}\\ \text{then } f \text{ diffeo}\end{array}
\end{array}$$

1.1. Diffeomorphisms.

Def. *A smooth map $f : M^m \to N^n$ is a* diffeo *if*

$$\begin{cases} f \text{ is bijective} \\ f^{-1} \text{ is smooth} \end{cases}$$

f is a local *diffeo if $\exists$ open $p \in U \subset M$ such that*

$$\begin{cases} f(U) \text{ is open} \\ f|_U : U \to f(U) \text{ is a diffeo} \end{cases}$$

Rmk. *This forces $m = n$ since*

$$f \text{ local diffeo at } p \Rightarrow d_pf \text{ isomorphism with inverse } d_{f(p)}f^{-1}$$

Thm (Inverse Function Theorem)**.**

$$d_pf \text{ isomorphism } \Rightarrow f \text{ local diffeo at } p$$

[14] at $(p,v) \neq (p,0)$ the iso $T_{(p,v)}E \cong T_pM \oplus E_p$ depends on ∇: ∇ decides which vectors are horizontal. *Non-examinable:* $T^{\mathrm{horiz}}_{(p,v)}E = ds_p(T_pM)$ for local sections s with $(\nabla_X s)_p = 0, \forall X \in T_pM$.

Ideas in the proof. [15]

$$\text{“}x = f^{-1}(y)\text{”} \Leftrightarrow y = f(x) \Leftrightarrow \text{fixed point } g_y(x) = x$$

where we define

$$g_y(x) = x + A^{-1}(y - f(x))$$

where $A = d_p f$ in local coords on a closed ball around p (which is a complete metric space). By the contraction mapping theorem[16] there exists f^{-1}, and one easily checks that f^{-1} is continuous. Now observe the following equation holds:

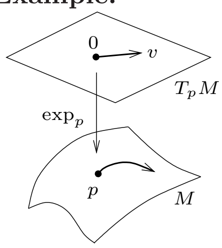
$$(df^{-1})(y) = [df(f^{-1}(y))]^{-1}$$

so by **bootstrapping**,[17]

$$\text{smoothness of } f^{-1} = \text{ smoothness of } f. \quad \square$$

Rmk. *The proof only used completeness and linear algebra, we never used compactness or finite dimensionality.*

Example.[18]



$d_0(\exp_p) \cdot \vec{v} = \partial_t|_{t=0} \exp_p(t\vec{v}) = \vec{v}$, *so* $d_0 \exp_p = I$,
$\Rightarrow \exp_p$ *is a local diffeo near* 0,
$\Rightarrow \exists \varepsilon > 0$ *so that* $\exp_p : D_\varepsilon T_p M \to M$ *is a chart!*
$\Rightarrow$ *This proves the uniqueness of geodesics joining two close enough points.*

1.2. **Regular maps.**[19]

Def. $d_p f$ *surjective* $\Rightarrow f$ *is* regular *at* p (submersion), *so* $m \geq n$.
$d_p f$ *not surjective* $\Rightarrow p$ *is a* critical point, $f(p)$ *is a* critical value.

Def. {regular values} $= N \setminus$ {critical values}. Note f may not attain these values![20]

Example. $f : M \to \mathbb{R}$ regular at $p \Leftrightarrow d_p f \neq 0$, so $\mathrm{Crit}(f) = \{p \in M : d_p f = 0\}$.

Thm (Implicit Function Theorem)**.**
f regular at $p \Rightarrow \exists$ local coords near $p, f(p)$ in which f is a projection:

$$f(x_1, \ldots, x_n, \ldots, x_m) = (x_1, \ldots, x_n)$$

Pf. Locally $f : \mathbb{R}^m \to \mathbb{R}^n$ defined near $p = 0$, $f(p) = 0$.
By linear algebra (row/col operations):

$$B \circ d_p f \circ A = (I \quad 0)\,,$$

[15] For a detailed proof, see for example Lang, *Undergraduate analysis*.

[16] *Contraction Mapping Theorem*: a contraction mapping on a complete metric space has a unique fixed point. *Contraction mapping* $f : M \to M$ means there is a constant $0 < k < 1$ for which $\mathrm{dist}(fx, fy) \leq k \cdot \mathrm{dist}(x, y)$, for all $x, y \in M$. *Fixed point* means a point x with $f(x) = x$.

[17] f^{-1} is cts $\Rightarrow df^{-1}$ cts (because of the equation) $\Rightarrow f^{-1}$ is once ctsly diffble $\Rightarrow$ can differentiate equation $\Rightarrow$ repeat. Note that we used that inversion is C^∞ on invertible matrices.

[18] by definition $u(t) = \exp_p(t\vec{v})$ satisfies $u'(t) = \vec{v}$.

[19] Motivation: what if $m \neq n$? Hwk 3 is about $m \leq n$, immersive maps. Here we do $m \geq n$.

[20] $r \in N$ is a *regular value* iff $d_p f$ is surjective for all $p \in f^{-1}(r)$. So this holds if $f^{-1}(r) = \emptyset$!

some $A \in GL(m), B \in GL(n)$. Replace f by $B \circ f \circ A$ (a linear change of coords), so $d_p f = (I \quad 0)$. Define

$$\begin{aligned} &F : \mathbb{R}^m \to \mathbb{R}^{n+(m-n)} = \mathbb{R}^m \text{ defined near } p \text{ by} \\ &F(x) = (f(x), x_{n+1}, \ldots, x_m) \end{aligned}$$

So $d_p F = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. So F local diffeo near p. The claim follows by the diagram:

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\text{old chart}"] \arrow[dd, bend right, "\text{new chart}"'] & N \arrow[dd, "\text{old chart}"] \\
\mathbb{R}^m \arrow[d, "F"] & \\
\mathbb{R}^m \arrow[r, "\text{projection}"] & \mathbb{R}^n
\end{tikzcd}$$

$\square$

Rmk. *diffeo/regularity are* open conditions*: if true at p then true near p.*

Rmk. *classically (algebraic geometry) you apply the theorem to*

$$F : \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n \times \mathbb{R}^{m-n}, \quad F(X, Y) = (f(X, Y), Y).$$

If $X \mapsto f(X, b)$ has non-singular derivative at $X = a$, and $f(a, b) = 0$, then near (a, b) the vanishing set

$$V(f) = f^{-1}(0) \text{ is parametrized by } Y \mapsto (0, Y) \overset{F^{-1}}{\mapsto} (g(Y), Y)$$

so $f(g(Y), Y) = 0$, and g is called implicit function.

Cor (Implicit Function Theorem).[21]

$$\boxed{q \in N \text{ regular value of } f \Rightarrow \begin{cases} f^{-1}(q) \subset M \text{ submfd of } \operatorname{codim} = n \\ T_p f^{-1}(q) = \ker d_p f \quad \forall p \in f^{-1}(q) \end{cases}}$$

Pf. Locally

$$\begin{aligned} f(x_1, \ldots, x_n, \ldots, x_m) &= (x_1, \ldots, x_n) \\ d_x f \cdot (\vec{x}_1, \ldots, \vec{x}_n, \ldots, \vec{x}_m) &= (\vec{x}_1, \ldots, \vec{x}_n). \end{aligned}$$

$\Rightarrow f^{-1}(0) = \{(0, \ldots, 0, x_{n+1}, \ldots, x_m)\}$, so those x's form local coords for chart.
$\Rightarrow T f^{-1}(0) = \ker df$. $\square$

[21] Codimension $f^{-1}(q) = n$ means $\dim f^{-1}(q) = m - n$. You lose n dimensions because you impose n conditions by asking $f = q$. The tangent space is also intuitive: if a curve moves along $f = q$, then f does not change, so $d_p f = 0$, and now recall that we related curves and vectors.