# LECTURE 4.

PART III, MORSE HOMOLOGY, 2011

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1.7. **Genericity.** Recall we defined: *almost every = full measure = generic.* Generic implies dense, but not conversely (e.g. $\mathbb{Q} \subset \mathbb{R}$ is dense but not generic).

**Thm** (Parametric Transversality)**.** *Let $M, N$ be closed mfds, $Q \subset N$ a submfd, and $S$ a mfd*[1] *without boundary but possibly non-compact. Suppose:*

$$F : M \times S \to N \text{ smooth map and } F \pitchfork Q$$

*Then $F_s = F(\cdot, s) \pitchfork Q$ for generic $s \in S$.*

*Proof.* Consider the projection $\pi$:

$$\begin{array}{ccc} M \times S \xrightarrow{F} N & \qquad & W = F^{-1}(Q) \xrightarrow{F|_W} Q \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ S & & S \end{array}$$

where we used $F \pitchfork Q$ to deduce $W = F^{-1}(Q)$ is a mfd.

**Claim.** $s \in S$ regular for $\pi|_W \Leftrightarrow F_s \pitchfork Q$.
(So the Thm follows by Sard applied to $\pi|_W$)
*Proof of Claim.* Suppose $q = F(m, s) \in Q$, so $w = (m, s) \in W$. $F \pitchfork Q$ implies:

$$(*) \qquad TN = dF \cdot T(M \times S) + TQ \qquad \text{at } F(w)$$

and it implies

$$\begin{aligned} T_w W &= \ker(T(M \times S) \overset{dF}{\to} TN \to TN/TQ) \qquad \text{at } w \\ &= \{(\vec{m}, \vec{s}) \in T_w(M \times S) = T_m M \oplus T_s S : dF \cdot \vec{m} + dF \cdot \vec{s} \in TQ\} \\ &= \{(\vec{m}, \vec{s}) : dF \cdot \vec{m} = -dF \cdot \vec{s} \text{ modulo } TQ\}. \end{aligned}$$

Finally, observe that

$$\begin{aligned} s \text{ regular for } \pi|_W &\Rightarrow d\pi|_W : TW \to TS \text{ surjective at } w \\ &\Rightarrow \forall \vec{s},\ \vec{s} = d\pi|_W \cdot (\vec{m}, \vec{s}) \text{ some } (\vec{m}, \vec{s}) \in T_w W \\ &\Rightarrow \forall \vec{s},\ dF \cdot \vec{m} = -dF \cdot \vec{s} \text{ modulo } TQ \text{ some } \vec{m} \\ &\overset{(*)}{\Rightarrow} \forall \vec{n} \in T_q N,\ \vec{n} = dF \cdot \vec{m}_2 + dF \cdot \vec{s} + \vec{q} \text{ some } \vec{m}_2, \vec{s}, \vec{q} \\ &\Rightarrow \forall \vec{n} \in T_q N,\ \vec{n} = dF \cdot \vec{m}_2 - dF \cdot \vec{m} \text{ modulo } TQ \\ &\Rightarrow TN = dF \cdot TM + TQ \text{ at } F(w) \qquad (**) \\ &\Rightarrow F_s \pitchfork Q \text{ at } w. \end{aligned}$$

The proof also works by reversing the implications, which proves the converse. $\square$

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[1] the *parameter space*.

**Modern viewpoint:** compute ker and coker of $d\pi|_W$ at $w = (m,s)$, $F(w) = q \in Q$:

$$\begin{aligned} \ker(d\pi|_W)_w &= \{(\vec{m},0) \in T_wW\} \\ &\cong \{\vec{m} \in T_mM : dF \cdot \vec{m} \in TQ\} \\ &= \ker\Big( \begin{tikzcd} T_mM \arrow[r, "dF=dF_s"] \arrow[rr, bend right, "DF=DF_s"'] & TN \arrow[r] & TN/TQ = \nu_Q \end{tikzcd} \Big) \end{aligned}$$

Therefore $\boxed{\ker(d\pi|_W)_w = \ker(DF_s : T_mM \to \nu_Q)}$ (which is $TF_s^{-1}(Q)$ if $F_s \pitchfork Q$).

Now consider $\mathrm{coker}(d\pi|_W)_w = T_sS/d\pi{\cdot}TW$, which you can think of as measuring how much the implication $* \Rightarrow **$ fails to hold. By linear algebra,[2]

$$\begin{aligned} & d\pi : \frac{TM \oplus TS}{TM + TW} \to \frac{TS}{d\pi \cdot TW} = \mathrm{coker}(d\pi|_W)_w \quad \text{iso at } w \\ F \pitchfork Q \quad \Rightarrow \quad & TW = \ker(DF : TM \oplus TS \overset{\text{surj}}{\to} \nu_Q) \quad \text{at } w \\ \Rightarrow \quad & \frac{TM \oplus TS}{TW} \to \nu_Q \quad \text{iso at } w \\ \Rightarrow \quad & \frac{TM \oplus TS}{TM + TW} \to \frac{\nu_Q}{DF \cdot TM} = \frac{\nu_Q}{DF_s \cdot TM} = \mathrm{coker} DF_s \quad \text{iso at } w \end{aligned}$$

So $\boxed{\mathrm{coker}(d\pi|_W)_w \cong \mathrm{coker}(DF_s : T_mM \to \nu_Q)}$.

These calculations only used linear algebra, so they hold also for Banach manifolds (which use a Banach space instead of $\mathbb{R}^n$ for charts, more on this in Lecture 5).

**Thm** (Parametric Transversality 2)**.**

$$\begin{aligned} F \pitchfork Q \quad &\Rightarrow \quad \begin{cases} \ker(d\pi|_W)_w = \ker(DF_s : T_mM \to \nu_Q) \\ \mathrm{coker}(d\pi|_W)_w \cong \mathrm{coker}(DF_s : T_mM \to \nu_Q) \end{cases} \\ &\Rightarrow \quad \begin{cases} d\pi \text{ Fredholm} \Leftrightarrow DF \text{ Fredholm}^{[3]} \\ d\pi \text{ surjective} \Leftrightarrow DF \text{ surjective} \end{cases} \end{aligned}$$

**Thm** (Genericity of transversality)**.** *Let $f : M \to N$ be smooth, $Q \subset N$ a submfd ($M, N, Q$ closed mfds). Then for $S =$open nbhd of $0 \in \mathbb{R}^k$, there is $F : M \times S \to N$, $F(\cdot, 0) = f$, with $F \pitchfork Q$.*

*Proof.* Embed $N \hookrightarrow \mathbb{R}^k$. Pick tubular nbhd of $N$: $U \subset \mathbb{R}^k$, $\pi : U \to N$. Then

$$\begin{array}{rcccccc} F: & M \times \mathbb{R}^k & \to & U & \to & N \\ & (m,s) & \mapsto & f(m) + s & \to & \pi(f(m)+s) \end{array}$$

the first map is defined for small $\|s\|$, and is clearly regular (think about it). The second map is regular by definition of $U$. Therefore the composite is regular. So $F \pitchfork$ anything (since $dF$ is already surjective), in particular $F \pitchfork Q$. $\square$

**Cor.**[4] *$f$ is homotopic to $f_s = F(\cdot, s) \pitchfork Q$ (for generic $s$).*

**Rmk.**

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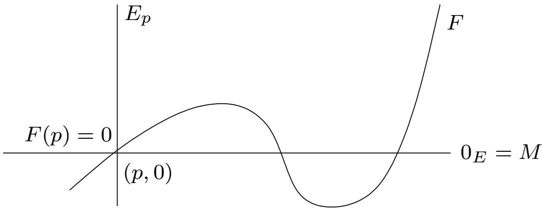
[2] $d\pi : TM \oplus TS \to TS$, $d\pi \cdot (\vec{m}, \vec{s}) = \vec{s}$ is surjective. Make it injective by quotienting the domain by $TM$. Now you want $TS/d\pi \cdot TW$ as codomain, so to make the map well-defined you quotient the domain by $TM + TW$.

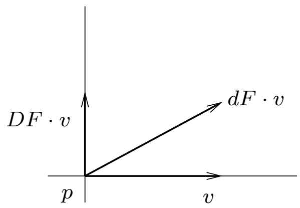
[3] Fredholm = finite dimensional kernel and cokernel, more on this in Lecture 6.

[4] Motto: You can make things transverse by perturbing!

(1) *You only need to perturb $f$ near $f^{-1}(nbhd(Q))$, indeed by replacing $s$ by $\beta(m)s$ where $\beta : M \to [0,1]$, $\beta = 1$ near $f^{-1}(Q)$, $\beta = 0$ away from $f^{-1}(Q)$, we still get regularity of $F$ near $F^{-1}(Q)$ so $F \pitchfork Q$.*
(2) *If $f$ is already $\pitchfork Q$ on a closed set $M_0 \subset M$ (hence near $M_0$ by openness of transversality), then one only needs to perturb $f$ away from $M_0$: again pick $\beta : M \to \mathbb{R}$, $\beta = 0$ on $M_0$, $\beta = 1$ away from $M_0$ (ensure $0 < \beta < 1$ lies in region where $f \pitchfork Q$, so for small enough $s$ also $f_s \pitchfork Q$ there).*
(3) *Instead of using $N \subset \mathbb{R}^k$ one can also use charts $U \subset N$, $\varphi : U \to \mathbb{R}^n$, and consider $F(m,s) = \varphi \circ f(m) + \beta(\varphi(m)) \cdot s$, $\beta =$bump function supported in chart. So one can inductively perturb $f$ on charts to make it $\pitchfork Q$.*

1.8. **Sections of a vector bundle.**




Recall $D_pF : T_pM \to E_p$ is the vertical derivative (vertical projection of $dF$).

**Lemma.** $\boxed{D_pF \textit{ surjective } \forall p \in F^{-1}(0_E) \Leftrightarrow F \pitchfork 0_E}$

*Proof.* $T_{(p,0)}E = T_p0_E \oplus E_p$, so $d_pF(T_pM) + T_p0_E = D_pF(T_pM) + T_p0_E$. □

**Cor.**

$$\boxed{DF \textit{ surjective along } F^{-1}(0_E) \Rightarrow \begin{cases} F^{-1}(0_E) \subset M \textit{ submfd} \\ \textit{of codim} = \textit{codim } 0_E = \textit{rank } E \\ TF^{-1}(0_E) = \ker DF \end{cases}}$$

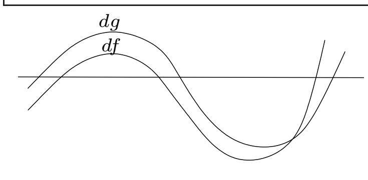
**Example.** *$E = T^*M \to M$ with section $F = df$, where $f : M \to \mathbb{R}$ smooth.*

1.9. **Morse functions.**

**Def.** *$f : M \to \mathbb{R}$ is a Morse function if* $\boxed{df \pitchfork 0_{T^*M}}$

**Consequences** (for $M$ closed):
(1) $df^{-1}(0_{T^*M}) = \mathrm{Crit}(f)$ is a 0-dim submfd, so the critical points are isolated, so $\boxed{\mathrm{Crit}(f) \text{ is finite}}$.
(2) $\boxed{f \text{ Morse} \Leftrightarrow \text{all critical pts are nondegenerate (Hessian is nonsingular)}}$

*Proof.* Hwk 2: at $p \in \mathrm{Crit}(f)$, $\mathrm{Hess}_p f = D_p(df) = \frac{\partial^2 f(p)}{\partial x_i \partial x_j}$. □
(3) Being Morse is stable.
(4) $\boxed{\text{Being Morse is open in the } C^2\text{-topology}}$




*Proof.* For Morse $f$:
$f, g$ $C^2$-close $\Rightarrow df, dg$ $C^1$-close
$\Rightarrow dg \pitchfork 0$ since $df \pitchfork 0$
$\Rightarrow g$ Morse. □

(5) $\boxed{\text{Morse functions are dense in the } C^0\text{-topology}}$

(Means: $\forall \varepsilon > 0, h : M \to \mathbb{R} \Rightarrow \exists$ Morse $f : M \to \mathbb{R}, \sup |f - h| < \varepsilon$)

*Proof.*[5] WLOG[6] $M \subset \mathbb{R}^k$, $h : \mathbb{R}^k \to \mathbb{R}$ (extend to $\mathbb{R}^k$ via a tubular nbhd and bump function). WLOG $h$ smooth (since $C^\infty \subset C^0$ dense). For $q \in \mathbb{R}^n$,

$$L_q : \mathbb{R}^k \to \mathbb{R},\ L_q(x) = \langle q, x\rangle_{\mathbb{R}^k} = \sum q_i \cdot x_i$$

is called a *height function.*

**Claim.** $h + L_q$ is Morse for almost every $q$ (and $C^0$-close to $h$ for small $q$)

*Proof.* Consider $F(x, q) = d(h + L_q)$:

$$\begin{tikzcd} M \times \mathbb{R}^k \arrow[r, "F"] \arrow[d, "\pi"] & T^*M \\ \mathbb{R}^k & \end{tikzcd}$$

We want $F \pitchfork 0_{T^*M}$, then $d(h+L_q) \pitchfork 0_{T^*M}$ for generic $q$ ✓. View its vertical component $F^{loc}$ as a map $\mathbb{R}^k \times \mathbb{R}^k \to T^*_x\mathbb{R}^k$ (later restrict to $M \subset \mathbb{R}^k$):

$$F^{loc}(x, q) = \textstyle\sum_i (\frac{\partial h}{\partial x_i}(x) + q_i)\, dx_i$$
$$DF_{(x,q)} \cdot (\vec{x}, \vec{q}) = \textstyle\sum_i (\sum_j \frac{\partial^2 h}{\partial x_j \partial x_i} dx_j(\vec{x}) + dq_i(\vec{q}))\, dx_i$$

*Key remark:* $dq_i(\vec{q})$ is arbitrary as you vary $\vec{q} \in T_q\mathbb{R}^k$. Now restrict:

$$DF_{(x,q)} : T_xM \times T_q\mathbb{R}^k \xrightarrow{F} T^*_x\mathbb{R}^k \xrightarrow{\text{pullback}} T^*_xM$$

The first map is surjective by the Key remark (can still freely vary $\vec{q}$), the second map is surjective because $M \hookrightarrow \mathbb{R}^k$ is embedded so $T_xM \hookrightarrow T_x\mathbb{R}^k$ is injective so its dual is surjective. So $DF_{(x,q)}$ surjective, so $F \pitchfork 0$ $\square$

**Cor.** *Almost any height function on $M \subset \mathbb{R}^k$ is Morse (take $h = 0$).*

(6) **Morse Lemma**

$$\boxed{f \text{ Morse} \Leftrightarrow \begin{cases} \exists \text{ local coords near each crit point } p \text{ (called \textit{Morse chart})} \\ \text{such that } f(x) = f(p) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_m^2 \end{cases}}$$

*Proof.* See Hwk 4. Key idea: Taylor $f(x) = f(p) + \frac{1}{2}\sum A_{ij}(x)x_ix_j$ with $A(x)$ symmetric. Diagonalize $A(x)$ smoothly in $x$. Then rescale coords. $\square$

**Def.** *The* Morse index *of $p \in Crit(f)$ is the index $i$ in the Morse Lemma:*

$$\boxed{|p| = ind_f(p) = i = \#(\textit{negative evalues of } Hess_p(f) \textit{ in local coords})}$$

*which equals the dimension of the maximal vector subspace of $T_pM$ on which $T_pM \otimes T_pM \to \mathbb{R}$, $(v, w) \mapsto D_p(df) \cdot (v, w)$ is negative definite.*[7]

(7) $\boxed{\text{Morse functions are generic}}$ *Proof.* Hwk 6.

[5]A messier alternative (avoiding $M \hookrightarrow \mathbb{R}^k$, and works for noncompact $M$): inductively perturb $f$ on charts by adding $\phi_j(x) \cdot L_{q_j}(x)$, where $\phi_j$ is a partition of unity subordinate to a countable locally finite cover by charts, $q_j$ are generic and small chosen inductively so that $f$ stays Morse on charts where you already perturbed.

[6]Without Loss Of Generality.

[7]$D_p(df) : T_pM \to T^*_pM$, so $D_p(df)$ eats two vectors and outputs a number.