LECTURE 5.

PART III, MORSE HOMOLOGY, 2011

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Motivation. $\mathcal{M}(p,q) = \{-\nabla f \text{ flowlines } u : \mathbb{R} \to M\}$ is an infinite dimensional manifold. What does this mean?

2. Banach spaces and manifolds

2.1. Review of Banach spaces.

Def. *Banach space = complete*[1] *normed vector space B. Call $\|\cdot\| : B \to \mathbb{R}$ the norm.*

Fact. *The unit ball in a Banach space is compact $\Leftrightarrow$ B is finite dimensional.*

Examples.
- $C^0[0,1] = \{\text{continuous } [0,1] \to \mathbb{R}\}$,

$$\|f\|_\infty = \max |f| \qquad (\text{"uniform norm"})$$

- $C^k[0,1] = \{k\text{-times ctsly diffble } [0,1] \to \mathbb{R}\}$,

$$\|f\|_{C^k} = \sum_{0 \le j \le k} \|D^j f\|_\infty \qquad \left(D = \frac{d}{dx}, D^0 f = f\right)$$

Arzela-Ascoli theorem.[2] *K compact metric space, $F \subset C(K) = \{cts\ K \to \mathbb{R}\}$ equibounded*[3] *and equicontinuous,*[4] *then F is precompact*[5] *using $\|\cdot\|_\infty$.*

Non-examples.
- $C_c(\mathbb{R}) = \{\text{compactly supported}$[6] $\text{cts } f : \mathbb{R} \to \mathbb{R}\}$ with $\|\cdot\|_\infty$.
- $C^0[0,1]$ with

$$\|f\|_p = \left(\int_0^1 |f(t)|^p \, dt \right)^{1/p} \qquad (1 \le p < \infty)$$

- $C^\infty[0,1]$ with the *natural topology*[7]: the topology is generated by all $U \cap C^\infty[0,1]$ with U open in $(C^k[0,1], \|\cdot\|_{C^k})$ some $k \ge 0$. Explicitly:

$$f_n \to f \text{ in } C^\infty \Leftrightarrow \|f_n - f\|_{C^{k(n)}} \to 0 \text{ some } k(n) \to \infty.$$

Lemma. *$C^\infty[0,1]$ is a complete metric space, but not a Banach space.*

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[1] *Complete* = Cauchy sequences converge: "$\|x_n - x_m\| \to 0$ as $n, m \to \infty$" implies "$\exists\, x : x_n \to x$".

[2] if you are curious, see Rudin, *Functional Analysis*.

[3] $\sup\{|f(x)| : f \in F\} < \infty$ for any given $x \in K$.

[4] $\forall \varepsilon > 0, x \in K$, there is a nbhd U of x such that $|f(y) - f(x)| < \varepsilon$ for all $y \in U, f \in F$.

[5] the closure of F in $C(K)$ is compact, explicitly: any sequence $f_n \in F$ has a uniformly convergent subsequence (but the limit may not be in F).

[6] $f = 0$ outside a compact. Equivalently: the support $\mathrm{supp}(f) = \overline{\{x : f(x) \neq 0\}}$ is compact.

[7] **Motivation:** we want $\{f : \|f\|_{C^k} < \varepsilon\}$ to be open for all $k \ge 0, \varepsilon > 0$. Asking that f, g are close in C^∞ iff *all* their derivatives are within ε is too harsh as there would be very few C^∞-functions considered "close to" 0: essentially only Gaussian functions like $\varepsilon(1 - \exp(-1/x^2))$.

Proof.
- Metric: $d(f,g) = \max_k \frac{2^{-k}\|f-g\|_{C^k}}{1+\|f-g\|_{C^k}}$
- Cauchy $\Rightarrow$ uniform convergence of any derivative (since $[0,1]$ compact)
  $\Rightarrow$ C^∞-convergence
  $\Rightarrow$ completeness.
- *Claim.* closed & bounded subsets F are compact

Pf. bounded means $\|f\|_{C^k} < K_k$ some constants K_k.
  $\Rightarrow D^{k-1}f$ equicts (by mean value theorem)
  $\Rightarrow$ a sequence in F has subseq f_n with $D^{k-1}f_n$ cgt in C^0 (by Arzela-Ascoli)
  $\Rightarrow$ a sequence in F has subseq f_n with $D^k f_n$ cgt in C^0, $\forall k$ (diagonal argument)
  $\Rightarrow$ a sequence in F has subseq f_n cgt in C^∞. ✓
- Suppose C^∞ is Banach (= $\exists$ norm inducing above topology). Then the closed unit ball is compact (by Claim) so C^∞ is finite dimensional (by Fact). Contradiction! □

Completion. From a normed space $(V, \|\cdot\|_V)$ we can produce a Banach v.s. B:

$$B = \{\text{Cauchy sequences } x_n \in V\} / {\scriptstyle (x_n)\sim(y_n) \text{ iff } \atop \scriptstyle \|x_n - y_n\|_V \to 0}$$

Define $\|x_n\|_B = \lim_{n\to\infty} \|x_n\|_V$. Get dense isometric inclusion $V \to B, x \mapsto (x_n = x)$.

Examples.
- $L^p[0,1]$ = completion of $(C^0[0,1], \|\cdot\|_p)$ $\quad (1 \le p < \infty)$
- $W^{k,p}[0,1]$ = completion of $(C^k[0,1], \|\cdot\|_{k,p})$,

$$\|f\|_{k,p} = \sum_{0\le j\le k} \|D^j f\|_p.$$

Fact. *C^∞ is dense in C^0, C^k, so can replace C^0, C^k by C^∞ above.*

Rmk. *$p = 2$ is particularly useful because it is Hilbert[8] using $\langle f,g\rangle_{L^2} = \int_0^1 f(x)g(x)\,dx$ on $L^2[0,1]$. Having a notion of perpendicularity goes a long way.*

Rmk. *Since C^∞ is not Banach, you must pass to $C^k, L^p, W^{k,p}$ to use big theorems (inverse fn thm, etc.). Big issue: how to recover C^∞ results e.g. from $C^k\, \forall k$?*

Trick: passing from $C^k, \forall k$ to C^∞: Suppose $\mathcal{M}_k \subset C^k = C^k[0,1]$ dense & open $\forall k$, and $\mathcal{M}_{k+1} = \mathcal{M}_k \cap C^{k+1}$. *Claim.* $\mathcal{M}_\infty = \cap \mathcal{M}_k \subset C^\infty$ dense & open.

Proof. • $f \in C^\infty \Rightarrow f \in C^k \Rightarrow \exists f_k \in \mathcal{M}_k : \|f_k - f\|_{C^k} < \varepsilon/2$
- $C^\infty \subset C^k$ dense $\Rightarrow \exists g_k \in C^\infty : \|f_k - g_k\|_{C^k} < \varepsilon/2$
- $\mathcal{M}_k \subset C^k$ open $\Rightarrow$ can ensure $g_k \in \mathcal{M}_k$, so $g_k \in \mathcal{M}_k \cap C^\infty = \mathcal{M}_\infty$

Conclusion: $\mathcal{M}_\infty \subset C^\infty$ is C^k-dense.
- $f \in C^\infty \Rightarrow \exists f_k \in \mathcal{M}_\infty : \|f - f_k\|_{C^k} < 1/k \Rightarrow f_k \to f$ in C^∞

Conclusion: $\mathcal{M}_\infty \subset C^\infty$ is C^∞-dense.
- $\mathcal{M}_\infty \subset C^\infty$ open since $\mathcal{M}_\infty = \mathcal{M}_k \cap C^\infty$ and $\mathcal{M}_k \subset C^k$ is C^k-open. □

2.2. **Banach manifolds.**

Def. *A (smooth)* Banach mfd X *modeled on the Banach space B is a Hausdorff, second-countable[9] topological space together with:*

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[8] Banach space with inner product such that $\|b\|^2 = \langle b, b\rangle$.

[9] *Second-countable*= there exists a countable basis for the topology. Consequence 1: any cover has a countable subcover. Consequence 2: it is *separable* = there is a countable dense subset.

- *open covering* $X = \cup U_i$*;*
- *charts: homeomorphisms* $\varphi_i : U_i \to$ *open* $\subset B$*;*
- *transitions:* $\varphi_j \circ \varphi_i^{-1}$ *are* C^k*-differentiable* $\forall k$*.*

If the transitions are only C^k*, then it's a* C^k*-Banach mfd.*

Rmk. $\psi : B \to B$ *is* differentiable *at* x *if* $\exists$ ***bounded***[10] *linear* $L : B \to B$ *with* $\psi(x+y) = \psi(x) + L \cdot y + \mathcal{o}(\|y\|)$*, where* $\mathcal{o}(t)$ *is an error term satisfying* $\lim\limits_{t\to 0} \frac{\mathcal{O}(t)}{t} = 0$.

Warning: (smooth) partitions of unity do not exist in general.

Non-examinable Example.
Claim. M, N closed mfds $\Rightarrow X = C^k(M,N)$ is a C^∞-Banach mfd.

- Avoiding charts: embed $N \subset \mathbb{R}^k$, pick a smooth retraction $\pi : U \to \mathbb{R}^k$ of an open tubular nbhd U of N. Observe: $X \subset C^k(M,U)$ is closed, $C^k(M,U) \subset C^k(M,\mathbb{R}^k)$ is open, $C^k(M,\mathbb{R}^k)$ is a Banach space. Since

$$\pi\circ : C^k(M,U) \to C^k(M,N) = X$$

is a smooth retraction, 1.6 Remark (2) $\Rightarrow X$ is a Banach submfd of $C^k(M,\mathbb{R}^k)$.

- Charts: we obtain nearby maps by the geodesic flow. Details: fix a Riemannian metric on N, fix a *smooth* $f : M \to N$. Then C^k-maps close to f are parametrized by[11] $C^k(D_\varepsilon f^*TN)$. Indeed the chart is $C^k(D_\varepsilon f^*TN) \to C^k(M,N), v \mapsto f_v$ where

$$f_v(p) = \exp_{f(p)} v(p).$$

One can[12] check $C^k(f^*TN)$ is a Banach space for the norm $\|v\|_{C^k} = \sum_{j\le k} \sup |\nabla^j v(p)|$. This gives a local chart[13] near f. They cover X since smooth f are dense in X.

- Transitions: $f, g \in X$ smooth & close $\Rightarrow C^k(f^*TN) \to C^k(g^*TN)$ on overlap[14] is

$$T : v \mapsto \exp_{f(p)} v(p) \mapsto \exp_{g(p)}^{-1}(\exp_{f(p)} v(p))$$

Key observation: differentiating in v means $d_vT \cdot \vec{v} = \partial_s|_{s=0} T(v + s\vec{v})$, for $\vec{v} \in C^k(f^*TN)$. This involves derivatives of exp, g, f: these are all smooth![15]

- Second countable: any $f \in C^k(M,N)$ can be approximated by a smooth map, which can be extended to a smooth map $\mathbb{R}^l \to \mathbb{R}^k$ after embedding $M \subset \mathbb{R}^l, N \subset \mathbb{R}^k$. Now approximate that by a map $\mathbb{R}^l \to \mathbb{R}^k$ whose coordinates are polynomials in l variables.[16] Approximate those polynomials by polynomials with $\mathbb{Q}$ coefficients. Hence X can be covered by charts as above constructed for restrictions to M of such rational polynomial maps. That is a countable basis for the topology.

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[10] *Bounded* means $\|L\| = \sup\limits_{y\neq 0\in B} \frac{\|Ly\|}{\|y\|} < \infty$. Easy fact: L continuous $\Leftrightarrow$ L bounded.

[11] The pull-back f^*TN is just the bundle over M with $(f^*TN)_p = T_{f(p)}N$ at $p \in M$. So $v \in C^k(D_\varepsilon f^*TN)$ means $v(p) \in T_{f(p)}N$, $|v(p)| \le \varepsilon$, and v is C^k.

[12] We'll see this norm in detail in future lectures. For now, think "manifold version" of $\|\cdot\|_{C^k}$.

[13] *Technical Rmk:* $B = C^k(f^*TN)$ is the same Banach space up to iso if you vary f by a smooth homotopy. If you change the homotopy class of f then B can change drastically. Indeed $C^k(M,N)$ is a disjoint union of connected components, each component is a Banach mfd consisting of homotopic maps. We are just using a different local model B for each component.

[14] If we wanted to use the same model $B = C^k(f^*TN)$, then we could always compose with the isomorphism induced by the diffeo $g^*TN \cong f^*TN$ obtained from the exponential map.

[15] Do not confuse this with the unrelated local derivatives $\partial_i v \in C^{k-1}$, which never arise!

[16] This can be done by the Stone-Weierstrass theorem.

Upshot: can now redo differential geometry: tangent vectors, derivative map, etc.

General principle: Results involving linear algebra or inverse function theorem generalize to Banach manifolds. But anything which involved compactness may fail unless you first reduce the problem to finite dimensional submfds/subspaces.

What works:
- Inverse function theorem (IFT) ✓
- Implicit function theorem ✓ provided $\ker d_p f$ has a *closed* complement. $(*)$

(locally $f : B_1 \to B_2$, want to apply IFT to $F : B_1 = \ker d_p f \oplus C \to B_2 \oplus C$, $F(x, y) = f(x, y) \oplus y$, so need $B_2 \oplus C$ Banach, so need C closed)
- Transversality definitions, all results which did not use Sard, assuming $(*)$ ✓

What fails:
- Sard's theorem: but this has a chance at working if the map is an isomorphism up to finite dimensional errors.

Def. *A map $f : M \to N$ between Banach mfds is* Fredholm *if $d_p f : T_p M \to T_{f(p)} N$ is a Fredholm operator, meaning $d_p f$ has finite dimensional kernel and cokernel.*

Basic example of Fredholm operators.
$B =$ all sequences of reals $(r_1, r_2, r_3, \dots)$ with finite norm $\|r\| = \sum |r_i|$ (or $(\sum |r_i|^p)^{1/p}$)

Right shift $R : (r_1, r_2, \dots) \mapsto (0, r_1, r_2, \dots)$.
Left shift $L : (r_1, r_2, \dots) \mapsto (r_2, r_3, \dots)$.

These are Fredholm: $\ker R = 0$, $\operatorname{coker} R \cong \mathbb{R}$, $\ker L \cong \mathbb{R}$, $\operatorname{coker} L = 0$.