LECTURE 6.

PART III, MORSE HOMOLOGY, 2011

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2.3. **Fredholm theory.**

Def. *A bounded linear map* $L : A \to B$ *between Banach spaces is a* Fredholm operator *if* $\ker L$ *and* $\operatorname{coker} L$ *are finite dimensional.*

Def. *A map* $f : M \to N$ *between Banach mfds is a* Fredholm map *if* $d_p f : T_pM \to T_{f(p)}N$ *is a Fredholm operator.*

Basic Facts about Fredholm operators

(1) $K = \ker L$ *has a closed complement* $A_0 \subset A$.
(so the implicit function theorem applies to Fredholm maps).
Pf. pick basis $v_1, \ldots, v_k$ of K, pick dual[1] $v_1^*, \ldots, v_k^* \in A^*$. $A_0 = \cap \ker v_i^*$. $\square$

(2) $\operatorname{im}(L) = image(L) \subset B$ *is closed.*
(so $\operatorname{coker} L = B/\operatorname{im}(L)$ is Banach)
Pf. pick complement C to $\operatorname{im}(L)$. C is finite dim'l, so closed, so Banach.

$$\Rightarrow \mathcal{L} : A/K \oplus C \to B, L(\overline{a}, c) = La + c$$

is a bounded linear bijection, hence an iso (open mapping theorem). So $\mathcal{L}(A/K) = \operatorname{Im}(L)$ is closed. $\square$

(3) $A = A_0 \oplus K$, $B = B_0 \oplus C$ where $B_0 = \operatorname{im}(L)$, $C =$ complement ($\cong \operatorname{coker} L$).

$$\Rightarrow L = \left[\begin{smallmatrix} \text{iso} & 0 \\ 0 & 0 \end{smallmatrix}\right] : A_0 \oplus K \to B_0 \oplus C$$

Def. **index**$(L) = \dim\ker L - \dim\operatorname{coker} L$.

(4) *Perturbing* L *preserves the Fredholm condition and the index:*
Claim.[2] $s : A \to B$ bdd linear with small norm $\Rightarrow \exists$ "change of basis" isos

$$\begin{matrix} i : A \cong B_0 \oplus K \\ j : B \cong B_0 \oplus C \end{matrix} \quad \text{such that } j \circ (L+s) \circ i = \left[\begin{smallmatrix} I & 0 \\ 0 & \ell \end{smallmatrix}\right]$$

for some linear map $\ell : K \to C$. Note: $\dim\ker$ drops by $\operatorname{rank}(\ell)$, but also $\dim\operatorname{coker}$ drops by $\operatorname{rank}(\ell)$. So $\operatorname{index}(L) = \operatorname{index}(L+s)$.
Proof. $s = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, $L = \left[\begin{smallmatrix} T & 0 \\ 0 & 0 \end{smallmatrix}\right]$ (where T is an iso). So:

$$\begin{bmatrix} I & 0 \\ -c(T+a)^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} T+a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} I & -(T+a)^{-1}b \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -c(T+a)^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} T+a & 0 \\ c & -c(T+a)^{-1}b+d \end{bmatrix}$$
$$= \begin{bmatrix} T+a & 0 \\ 0 & -c(T+a)^{-1}b+d \end{bmatrix}$$

where we use that $(T+a)^{-1}$ is defined for small $\|s\|$:

$$(T+a)^{-1} = [T(I + T^{-1}a)]^{-1} = (I - T^{-1}a + (T^{-1}a)^2 - (T^{-1}a)^3 + \cdots)T^{-1}$$

that power series converges provided $\|T^{-1}a\| < 1$, which we guarantee by: $\|T^{-1}a\| < 1 \Leftarrow \|T^{-1}\| < \|a\|^{-1} \Leftarrow \|T^{-1}\| < \|s\|^{-1} \Leftarrow \|s\| < \|T^{-1}\|^{-1}$ (since $\|s\| \geq \|a\|$).

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[1] $v_i^*(v_j) = \delta_{ij}$, the v_i^* exist by the Hahn-Banach theorem.

[2] Claim implies $\dim\ker L$ is upper semicontinuous: $\dim\ker(L+s) \leq \dim\ker L$, for small $\|s\|$.

Cor. *M connected, f Fred map $\Rightarrow$ index*$(f) = \operatorname{index} d_p f$ *is indep of* $p \in M$.

2.4. **Sard-Smale Theorem.**

$f : M \to N$ smooth Fred map $\Rightarrow$ $\{$*regular values of* $f\} \subset N$ is a *Baire set.*

Baire set is a geometer's analogue of "full measure" or "generic" for Banach mfds.

Def. *$S \subset N$ is a* Baire set[3] *if S contains a countable intersection of open dense sets.*

Baire category thm. A Baire set in a complete metric space[4] is dense.

Proof of Sard-Smale.

Claim 1. $\exists$ *charts* $\substack{M \supset U \hookrightarrow A \cong B_0 \oplus K \\ N \supset V \hookrightarrow B \cong B_0 \oplus C}$ *such that locally* $f(b,k) = \begin{bmatrix} I & 0 \\ 0 & \ell(b,k) \end{bmatrix}$, *for some nonlinear* $\ell : B_0 \oplus K \to C$.

Pf. Centre the charts around $p \in U$, $f(p) \in V$. Take $K = \ker d_p f$, $C \cong \operatorname{coker} d_p f$. So $f : B_0 \oplus K \to B_0 \oplus C$, $f(0,0) = (0,0)$, $d_{(0,0)} f = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Implicit fn thm[5] $\Rightarrow$ (after a change of charts) $\boxed{f(b,k) = (b, \ell(b,k))}$ ✓

Claim 2. f is locally closed[6] *(indeed closed in the above charts).*

Pf. Suppose $f(b_n, k_n) \to (b,c)$, $(b_n, k_n) \subset$ bounded open $\subset B_0 \oplus K$. By Claim 1, $b_n \to b$. Now k_n bdd, K finite dim'l $\Rightarrow \exists$ cgt subseq $k_n \to k$. So $f(b,k) = (b,c)$ ✓

Claim 3. We can reduce to Sard's theorem:

From a cover by charts as above, pick[7] a countable subcover of M, so reduce to $f|_U : U \to V$. *Claim.* $V_{\text{reg}} = \{$*regular values of* $f|_U\} \subset V$ *is open and dense.*[8]

Pf. (critical points of $f|_U) \subset U$ is closed,[9] so by Claim 2, V_{reg} is open ✓

$$d_{(b,k)} f = \begin{bmatrix} I & 0 \\ * & d_{(b,k)} \ell|_K \end{bmatrix} \text{ surjective } \Leftrightarrow d_{(b,k)} \ell|_K \text{ surjective}$$

Note: $d_{(b,k)} \ell|_K = d_k(\ell_b)$ for $\ell_b : K \to C, k \mapsto \ell(b,k) \leftarrow$ map of finite dim'l spaces!
$\Rightarrow V_{\text{reg}} \cap (\{b\} \oplus C) = $ (reg. val's of $\ell_b) \subset \{b\} \oplus C \leftarrow$ dense inclusion by Sard!
$\Rightarrow V_{\text{reg}} \subset V$ dense ✓ $\square$

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[3]or *generic set*, or *residual set*. We often produce $S =$ countable intersection of dense opens.

[4] Banach mfds are (complete) metric spaces. *Non-examinable proof:* Urysohn's metrization theorem says every second-countable regular space is metrizable. Banach mfds are by definition second-countable. *Regular space* means given a point p not contained in a closed subset C, there exist disjoint open nbhds of p and C (for Banach mfds, take a chart centred at p, then consider the ε-radius open ball centre p and the complement of the 2ε-radius closed ball centre p).

[5]$f(b,k) = (\alpha(b,k), \beta(b,k))$. Inverse fn thm $\Rightarrow \exists$ local inverse to $h : B_0 \oplus K \to B_0 \oplus K$, $h(b,k) = (\alpha(b,k), k)$ near $(0,0)$. Hence $f \circ h^{-1}(b,k) = (b, \ell(b,k))$. $\square$

[6]locally, closed sets map to closed sets.

[7]Banach mfds are defined to be second-countable, hence Lindelöf (covers have ctble subcovers). *Non-examinable remark:* I want Banach mfds to be metric spaces (see footnote 4). For metric spaces: second-countable $\Leftrightarrow$ separable $\Leftrightarrow$ Lindelöf. As far as I know, if I replace second-countable by separable, then it's not clear Banach mfds are metric, so it's not clear Baire category applies.

[8]So the regular values of f is the intersection of the regular values of all $f|U$'s, so it's a countable intersection of open dense sets, as required.

[9]Regular points of any smooth map of Banach mfds form an open set: at regular p, $d_p f = [I\ 0]$ (after change of basis), so for q close to p, $d_q f = [T\ *]$ for some invertible T since invertibility is an open condition (which is proved by the power series argument as in (4) of 2.3).

Thm. *If $f : M \to N$ is a Fredholm C^k-map of C^k-Banach mfds, then Sard-Smale holds provided $k > \operatorname{index}(f)$.*

Proof. $l_b : K \to C$, $\operatorname{index}(f) = \dim K - \dim C$, now use C^k-Sard (see 1.3). ☐

Cor. *$F : M \times S \to N$ smooth map of Banach mfds, $Q \subset N$ submfd, $F \pitchfork Q$, such that $D_m F_s : T_m M \to \nu_{Q,F_s(m)}$ is Fredholm. Then $F_s \pitchfork Q$ for generic $s \in S$.*

Proof. Parametric transversality 1 & 2 (using Sard-Smale and Hwk 6). ☐

2.5. **Zero sets of Fredholm sections.**

Def. Banach vector bundle $\pi : E \to B$ *with fibre V, is defined analogously to finite dimensional vector bundles after replacing E, B by Banach mfds, and V by a Banach space.*

Thm. *For a Banach vector bundle $E \to M \times S$ and a smooth section $F : M \times S \to E$, assume for all (m, s) with $F(m, s) = 0$ that*

(1) $D_{(m,s)}F : T_{(m,s)}(M \times S) \xrightarrow{dF} T_{(m,s,0)}E \to E_{(m,s)}$ *is surjective*
(2) $D_m F_s : T_m M \to E_{(m,s)}$ *Fredholm*

Then, for generic $s \in S$,

$$\begin{cases} F_s^{-1}(0_E) \subset M \textit{ submfd of } \dim = \operatorname{index}(D_m F_s) \quad (\textit{near } m) \\ T_m F_s^{-1}(0_E) = \ker(D_m F_s : T_m M \overset{\text{surj}}{\longrightarrow} E_{(m,s)}) \end{cases}$$

Proof. This is a direct consequence of the Corollary, but since it's important:
(1) $\Rightarrow F \pitchfork 0_E \Rightarrow W = F^{-1}(0)$ mfd (implicit fn thm[10]).
Write $\pi : M \times S \to S$ for the projection, recall parametric transversality 2:

$$\ker d\pi|_W \cong \ker DF_s \qquad \operatorname{coker} d\pi|_W \cong \operatorname{coker} DF_s.$$

(2) $\Rightarrow d\pi|_W$ Fredholm of index = index DF_s.
Sard-Smale $\Rightarrow$ for generic s, $d\pi|_W$ is surjective along

$$\pi|_W^{-1}(s) = W \cap \pi^{-1}(s) = F_s^{-1}(0).$$

Hence DF_s is surjective (by the iso of cokernels above). So $F_s^{-1}(0)$ mfd and

$$TF_s^{-1}(0) \cong \ker DF_s$$

with $\dim T_m F_s^{-1}(0) = \dim \ker D_m F_s = \operatorname{index} D_m F_s$ (since $\operatorname{coker} D_m F_s = 0$). ☐

Thm. *Thm also holds for C^k-maps of C^k-Banach mfds when $k >$ index DF_s.*

Rmk. *The dimension of $F_s^{-1}(0_E)$ can vary depending on the connected component, since index$(D_m F_s)$ depends on the connected component of m. That is why we wrote "near m" in the Thm.*

[10] Hwk 6 checks the closed complement condition. You should check that Cor 1.2 (implicit function theorem) works also for Banach mfds.