LECTURE 7.

PART III, MORSE HOMOLOGY, 2011

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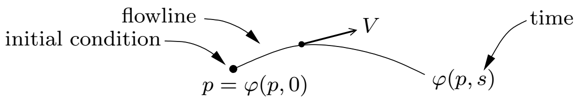
Motivation. Morse theory = studying the space of $\{-\nabla f$ flowlines $\}$.

3. Flowlines and Topology

3.1. **Flowlines.** M closed mfd, V smooth vector field.



Thm.
There exists a unique solution $\varphi : M \times \mathbb{R} \to M$ *of*

$$\frac{\partial \varphi}{\partial s} = V \circ \varphi \qquad \qquad \varphi(\cdot, 0) = id.$$

$\varphi_s = \varphi(\cdot, s) : M \to M$ *is a diffeo, with* $\varphi_s \circ \varphi_t = \varphi_{s+t}$.

Def. φ *is the* flow *of* V, *and* $s \mapsto \varphi(p,s)$ *is the* flowline *through* p.
By uniqueness, flowlines never intersect unless they coincide (up to $s \mapsto s + const$*).*

Proof. Locally: $y : [-\varepsilon, \varepsilon] \to \mathbb{R}^m$, $y(s) = \varphi(p,s)$ solves the ODE

$$y'(s) = V(y(s)) \qquad \qquad y(0) = p.$$

ODE theory[1] $\Rightarrow$ for small $\varepsilon > 0$, $\exists$ unique solution y which depends smoothly on the initial condition p.

Globally: $\forall p \in M$, local result yields a unique smooth map[2]

$$\varphi : U_p \times [-\varepsilon_p, \varepsilon_p] \to M \qquad \qquad (*)$$

Take a finite cover of M by U_p's, and ε = smallest of the ε_p's. So:

$$\varphi : M \times [-\varepsilon, \varepsilon] \to M \qquad \qquad (**)$$

Trick: $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for small s, t, since both solve $y(0) = \varphi_t(p)$, $y'(s) = V(y(s))$.

$\Rightarrow \varphi_s$ diffeo with inverse φ_{-s}

$\Rightarrow$ extend[3] φ_s to $s \in \mathbb{R}$: $\varphi_s = \varphi_{s/k} \circ \cdots \circ \varphi_{s/k}$

(k composites, with $k \gg 0$ so that $|s/k| < \varepsilon$) $\square$

Rmk. *ODE theory* $\Rightarrow$ *If* V *is* C^k *then* φ *is* C^k.
If M *is just a* C^k*-mfd and*[4] V *is* C^{k-1} *then* φ *is* C^{k-1}.

Rmk.

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[1] Lang, *Undergraduate Analysis*, or Lang, *Differential Manifolds*, prove this in great detail.

[2] pick the open nbhd U_p of p small enough, so that φ lands in the given chart.

[3] φ_s is well-defined, indeed: $(\varphi_{s/k'})^{k'} = (\varphi_{s/kk'})^{kk'} = (\varphi_{s/k})^k$.

[4] Since TM is just C^{k-1}, any higher differentiability of vector fields does not make sense.

(1) *If V_s depends on s, you pass to $M \times \mathbb{R}$, $V(p,s) = V_s(p) \oplus \frac{\partial}{\partial s}$. Since the mfd is now non-compact, $(*)$ holds but $(**)$ can fail (if $\varepsilon_{(p,s)} \to 0$ as $|s| \to \infty$).* ***Fact.*** *If V is C^1-bounded then $(**)$ holds.*[5] *So Thm holds.*
(2) *For non-compact mfd M or Banach mfd M, $(*)$ holds by the same proof, but for $(**)$ we need the condition:* $\exists K, R > 0$ such that $\forall p \in M$, $\exists$ chart $\varphi_p : U_p \to \mathbb{R}^m$ or B, such that V is C^1-bounded by K in chart and[6] $\varphi_p(U_p) \supset$ ball with centre $\varphi_p(p)$ radius R.

3.2. Negative gradient flowlines.

(M, g) closed Riemannian mfd. Write $|v| = g(v,v)^{1/2}$ for the norm.
$f : M \to \mathbb{R}$ smooth function.

Def. *The* gradient vector field ∇f *is defined by*

$$g(\nabla f, \cdot) = df$$

Locally:[7] $\nabla f = g^{-1}\partial f = \sum \partial_i f \cdot g^{ij} \cdot \partial_j$.

Rmk. $p \in \mathit{Crit}(f) \Leftrightarrow d_p f = 0 \Leftrightarrow (\nabla f)_p = 0 \Leftrightarrow |\nabla f|_p = 0$

For a $-\nabla f$ flowline $u : [a,b] \to M$ (so $u' = -\nabla f$) we care how f varies along u:

$$\partial_s(f \circ u) = df \cdot u' = df(-\nabla f) = g(\nabla f, -\nabla f) = -|\nabla f|^2$$
$$f(u(b)) - f(u(a)) = \textstyle\int_a^b \partial_s(f \circ u)\, ds = -\int_a^b |(\nabla f)_{u(s)}|^2\, ds \le 0$$

Def. *So it is natural to introduce the notion of* Energy *of a path $u : (a,b) \to M$:*

$$E(u) = \int_a^b |(\nabla f)_{u(s)}|^2\, ds \ge 0.$$

Note that $E(u) = 0$ iff u is constantly equal to a critical point.

Cor. *f decreases along $-\nabla f$ flowlines, and there is an a priori energy estimate:*[8] *for any $-\nabla f$ flowline from x to y,*

$$E(u) = f(x) - f(y).$$

In particular, $E(u)$ is a homotopy invariant relative to the ends.

Rmk (Novikov theory). *A generalization of Morse theory, called* Novikov theory*, replaces df by a closed 1-form α. This gives rise to a vector field via $g(V, \cdot) = \alpha$, and one studies $-V$ flowlines. The energy $E(u) = \int_a^b |V_{u(s)}|^2\, ds \ge 0$ is zero iff u is constantly equal to a zero of α. There is no a priori energy estimate. However, $E(u)$ is still a homotopy invariant of $-V$ flowlines relative to the ends. Indeed:*

$$E(u) = -\int_{[a,b]} u^*\alpha$$

[5] Why C^1? Locally it implies V is Lipschitz by the mean value theorem, which is what's needed to solve the ODE. C^1 bounds guarantee the Lipschitz constant is bounded uniformly.

[6] The C^1 bounds are calculated in a chart, but they can always be achieved by rescaling a chart. So the second condition is crucial (for example: consider $V = \frac{\partial}{\partial x}$ on $\mathbb{R} \setminus \{0\}$).

[7] g^{ij} = inverse matrix of $g_{ij} = g(\partial_i, \partial_j)$, $\partial_j = \frac{\partial}{\partial x_j}$.

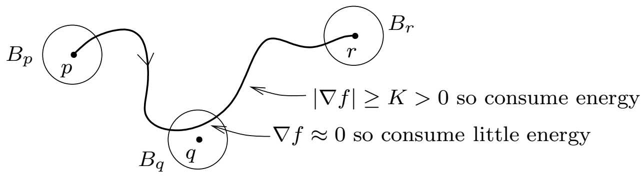
[8] *a priori* refers to the fact that the estimate only depends on boundary conditions x, y, not u.

since $u^*\alpha = \alpha \cdot u' = -\alpha(V) = -g(V,V) = -|V|^2$.
Proof: if $H : [a,b] \times [0,1] \to M$ *is a homotopy relative ends*[9]*, by Stokes's theorem*

$$\int_{[a,b]} u_0^*\alpha - \int_{[a,b]} u_1^*\alpha = \int_{[a,b]\times[0,1]} dH^*\alpha = 0$$

($dH^*\alpha = H^* d\alpha = 0$*, since* α *is closed). For* $\alpha = df$ *the energy estimate is Stokes:* $E(u) = -\int_{[a,b]} u^* df = -\int_{[a,b]} d(u^* f) = f(u(a)) - f(u(b))$ *for* $-\nabla f$ *flowlines* u*.*

3.3. **Energy consumption.**



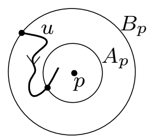
Lemma. *A* $-\nabla f$ *flowline from* x *to* y *landing in a region where* $|\nabla f| \geq K > 0$ *has*

$$E(u) \geq K \cdot \mathrm{dist}(x,y).$$

Pf.[10] Loosely:[11] $E(u) = \int |\nabla f|^2 \geq K \int |\nabla f| = K \int |u'| = K\,\mathrm{length}(u) \geq K \mathrm{dist}(x,y)$.

For example, this proof shows that: in the complement of small balls centred at the critical points of a Morse function f, any $-\nabla f$ flowline must consume at least some fixed amount $\delta > 0$ of energy to flow from one ball to another.

Notation. $A \subset\subset B$ (*compactly contained*) means: A, B open, and $A \subset \overline{A} \subset B$.



No escape Lemma. *Let* $p \in A_p \subset\subset B_p$ *with* $\overline{B_p} \cap Crit(f) = \{p\}$*. Then* $\exists \delta > 0$ *such that any* $-\nabla f$ *flowline needs* $E \geq \delta$ *to go from* ∂A_p *to* ∂B_p*, or vice-versa.*

Proof. Consider the region $\overline{B_p} \setminus A_p$, apply the Lemma. □

Energy quantum Lemma.[12] Pick disjoint nbhds $\overline{B_r}$ of each $r \in \mathrm{Crit}(f)$. $\exists \delta > 0$ such that any $-\nabla f$ flowline from B_p to B_q for $p \neq q$ consumes energy $E \geq \delta$.

Proof. Consider the region $M \setminus \cup \overline{B_r}$, apply the Lemma. □

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[9] $H(a,\cdot) = x$, $H(b,\cdot) = y$, $H(0,\cdot) = u_0$, $H(1,\cdot) = u_1$.

[10] This proof works similarly if we use α instead of df (see previous Rmk).

[11] *Distance* $\mathrm{dist}(x,y) =$ infimum of lenth$(u) = \int |u'(s)|\, ds$ over all curves u from x to y. **Rmk.** length is parametriz'n indep: $\int |(u \circ \phi)'(s))|\, ds = \int |u'(\phi(s))|\, \phi'(s)\, ds = \int |u'(s)|\, ds$ $(\phi'(s) > 0)$.

[12] Note: you could have $f(p) = f(q)$, so the energy estimate doesn't imply result immediately.

3.4. **Convergence at the ends.**

Thm. *For $f : M \to \mathbb{R}$ Morse, M closed mfd, any $-\nabla f$ flowline $u : \mathbb{R} \to M$ must converge at the ends to critical points, hence $\exists p, q \in Crit(f)$ with*

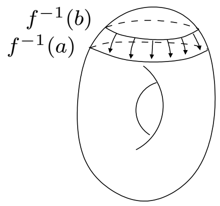
$$u \in W(p,q) = \{-\nabla f \textit{ flowlines } \mathbb{R} \to M \textit{ converging to } p, q \textit{ at } -\infty, \infty\}.$$

In particular, $W(p,p) = \{$constant flowline at $p\}$, since there $E = f(p) - f(p) = 0$.

Proof. [13] Case $s \to +\infty$ (for $s \to -\infty$ apply proof to $-f$). No Escape Lemma: for each $p \in \mathrm{Crit}(f)$ pick A_p, B_p's (small), get $\delta > 0$. $f \circ u$ decreases in s, but f is bounded (M compact), so $f \circ u \to r \in \mathbb{R}$, so for $s \gg 0$, $f \circ u$ is within δ of r.
Suppose $u \notin \cup B_p$ for some $s \gg 0$. Then u hasn't enough energy left to reach $\cup A_p$ for larger s. So $u \notin \cup A_p$ for $s \gg 0$. But $|\nabla f| \geq K > 0$ on $M \setminus \cup A_p$, so $f \circ u \to -\infty$, absurd. So $u \in \cup B_p$ for $s \gg 0$, and B_p is arbitrarily small. $\square$

3.5. **Topology of sublevel sets.**

$M_a = \{x \in M : f(x) \leq a\}$ are the *sublevel sets.*



Thm. *If $[a,b]$ contains no critical values of f, then $M_b \cong M_a$ are diffeo.*

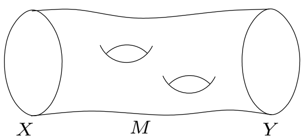
Proof. $\varphi =$ flow of $-\frac{\beta(f)}{|\nabla f|^2} \cdot \nabla f$, where $\beta : \mathbb{R} \to [0,1]$ is a bump function, $\beta = 1$ on $[a,b]$ and $\beta = 0$ on away from $[a,b]$ (in particular $\beta = 0$ at all critical values of f).

$$\partial_s(f \circ \varphi) = df \cdot \partial_s \varphi = g(\nabla f, -\tfrac{\beta(f)\nabla f}{|\nabla f|^2}) = -\beta(f)$$

which equals -1 on $[a,b]$. So $\varphi(\cdot, b-a) : M_b \to M_a$ diffeo. $\square$

Rmk. *The $-\nabla f$ flowlines are orthogonal to the regular level sets $f^{-1}(a)$.*
Pf. $g(-\nabla f, v) = -df \cdot v = 0$ *for* $v \in Tf^{-1}(a) = \ker df|_{f^{-1}(a)}$.

Rmk. *There is a deformation retraction*[14] *of M_b onto M_a: $r : M_b \times [0,1] \to M_b$, $r(x,s) = x$ if $x \in M_a$, and $r(x,s) = \varphi(x, s(f(x) - a))$ if $x \in f^{-1}[a,b]$.*



Def. *A* cobordism *between possibly-disconnected closed X^n, Y^n is a compact mfd M^{n+1} with $\partial M = X \sqcup Y$. Call it* h-cobordism *if in addition X, Y are deformation retracts of M.*

Fact.[15] *Equivalent definitions of h-cobordism:*

M *h-cobordism* $\Leftrightarrow$ ($X, Y \hookrightarrow M$ *hpy equivalences*) $\Leftrightarrow$ ($\pi_*(M,X) = \pi_*(M,Y) = 0$)

$\boxed{\text{For } X, Y, M \text{ simply connected: } (M \text{ h-cobordism}) \Leftrightarrow (H_*(M,X) = 0)}$

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[13] Curiosity: $\exists$non-insightful elementary proof by contradiction, avoiding energy arguments.

[14] *Deformation retraction* $r : X \times [0,1] \to X$ of X onto A means: r cts, $r|_A =$id, $r(X,1) = A$. Note r is a hpy from id_X to a retraction $r_1 = r(\cdot, 1)$ of X onto A (means $r_1(X) = A$, $r_1|_A = \mathrm{id}_A$).

[15] *Non-examinable:* By Whitehead's theorem and LES for relative hpy: inclusions $X, Y \hookrightarrow M$ are hpy equivalences $\Leftrightarrow$ they are isos on hpy gps $\Leftrightarrow \pi_*(M,X) = \pi_*(M,Y) = 0$. By hpy theory:[16] M deform retracts onto $X \Leftrightarrow \pi_*(M,X) = 0$. The 2nd fact uses Hurewicz: if $\pi_1(X) = 0$, $\pi_1(M,X) = 0$ then the first non-zero $\pi_k(M,X)$ is iso to the first non-zero $H_k(M,X)$; and it uses the Poincaré duality iso $H_*(M,X) \cong H^{m-*}(M,Y)$ (by universal coefficients, $H_*(M,Y) = 0 \Leftrightarrow H^*(M,Y) = 0$).

[16] Hilton, *An Introduction to Homotopy Theory*, Thm 1.7 p.98: if $\pi_*(Y, Y_0) = 0$, then $\forall$ subcx A of a CW cx X, any $(X,A) \to (Y,Y_0)$ can be homotoped to $X \to Y_0$ keeping it constant on A.

h-cobordism Thm[17] (Smale 1962) *If X, Y are simply connected of dim ≥ 5 then the h-cobordism M is trivial: $M \cong Y \times [0,1]$ diffeo. In particular $X \cong Y$ diffeo.*

Lemma. *If there exists a Morse function $f : M \to [a,b]$ with no critical points, $X = f^{-1}(a)$, $Y = f^{-1}(b)$, then M is a trivial h-cobordism.*

Proof. In notation of previous Rmk: $f^{-1}(b) \times [0,1] \to M, (x,s) \mapsto \varphi(x, s(b-a))$. □

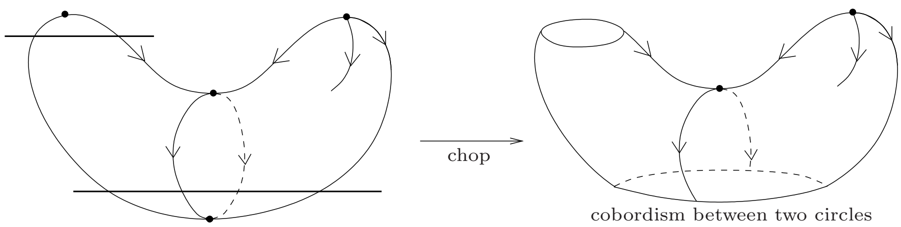
Cor. *An h-cobordism is trivial $\Leftrightarrow$ it admits a Morse function as in the Lemma.*

Idea of Pf of Thm.(hard!) Start with a Morse function on the cobordism. Systematically "cancel out" the crit points in pairs by locally modifying f and the flow, until there are no crit points left. Key: the use of ***gradient-like vector fields***: $V \in C^\infty(TM)$, $V(f) > 0$ (except at $\mathrm{Crit}(f)$), such that at each $p \in \mathrm{Crit}(f)$, $\exists$ Morse chart in which

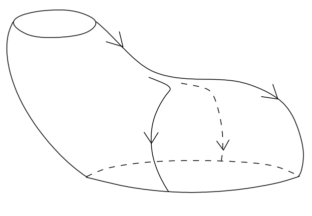
$$\begin{array}{rcl} f(x) & = & f(p) - x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_m^2 \\ V & = & -x_1\partial_1 - \ldots - x_i\partial_i + x_{i+1}\partial_{i+1} + \ldots + x_m\partial_m. \end{array}$$

The flow you consider is the flow of $-V$, not that of $-\nabla f$. This means you know exactly what v is near critical points: up to a constant, it is the Euclidean gradient in the Morse chart (whereas $-\nabla f$ is unknown since the metric in general is not Euclidean in the Morse chart!). The $V(f) > 0$ ensures that f still decreases along $-V$ flowlines, and you get good estimates of the energy $E = \int |V|^2$.

The idea behind cancelling out critical points in pairs is as follows. Consider the sphere with two discs cut out:



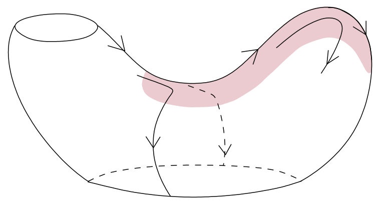
You would like to push down one of the hills:



In general, you do not know what the manifold looks like globally, so you actually just modify f, v locally near the trail going up the hill:[18]

[17] Standard great reference: Milnor, *Lectures on the h-cobordism theorem.*

[18] in the figure, we modify f, v in the shaded region.



Thus cancelling out the two critical points with index difference 1.