LECTURE 8.

PART III, MORSE HOMOLOGY, 2011

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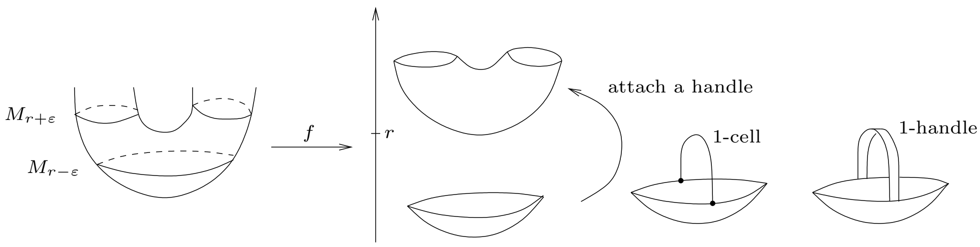
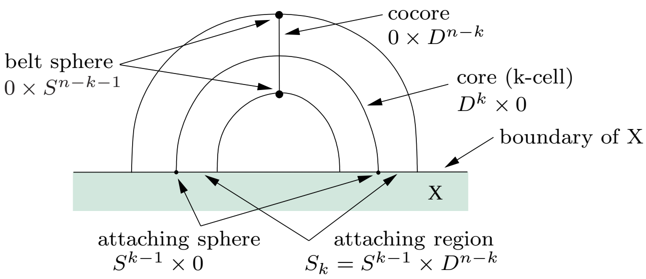


3.6. Handle attachments.

Def. *A* k-handle *is a "thickened up" k-cell:*[1] $D^k \times D^{n-k}$

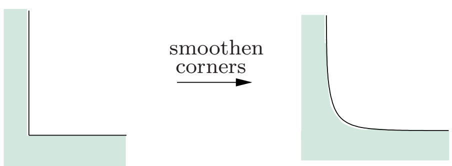


Attaching a k-handle *to* X^n *means:*

$$X \cup_{S_k} (D^k \times D^{n-k})$$

where the handle is attached along the attaching region $S_k = S^{k-1} \times D^{n-k}$ *via an embedding* $S_k \hookrightarrow \partial X$ *which needs to be specified.*

Rmk. *One can smoothen the corners of the attachment, but we omit the details.*



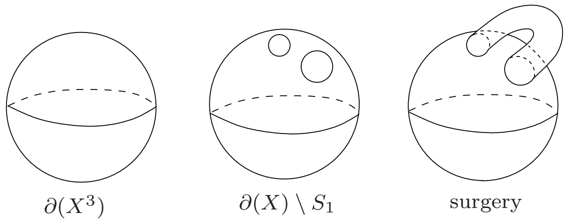
Date: May 3, 2011,

[1]where D^k is the closed unit disk in $\mathbb{R}^k$.

Rmk. *The boundary becomes:*

$$(\partial X \setminus \operatorname{int} S_k) \cup_{\partial S_k} (D^k \times S^{n-k-1})$$

called a surgery on $S_k \hookrightarrow \partial X$ *(or* surgery on ∂X*). Equivalently, it is a surgery on the knot* $S^{k-1} = S^{k-1} \times 0 \overset{\varphi}{\hookrightarrow} \partial X$ *with framing*[2]*. Note that this attachment kills the homotopy class of* $S_k \hookrightarrow \partial X$*. Example: surgery on a 3-sphere with a 1-handle:*



3.7. Topology of sublevel sets II.

Thm (Handle-attaching theorem)**.** M^m *closed,* $f : M \to \mathbb{R}$ *Morse.*

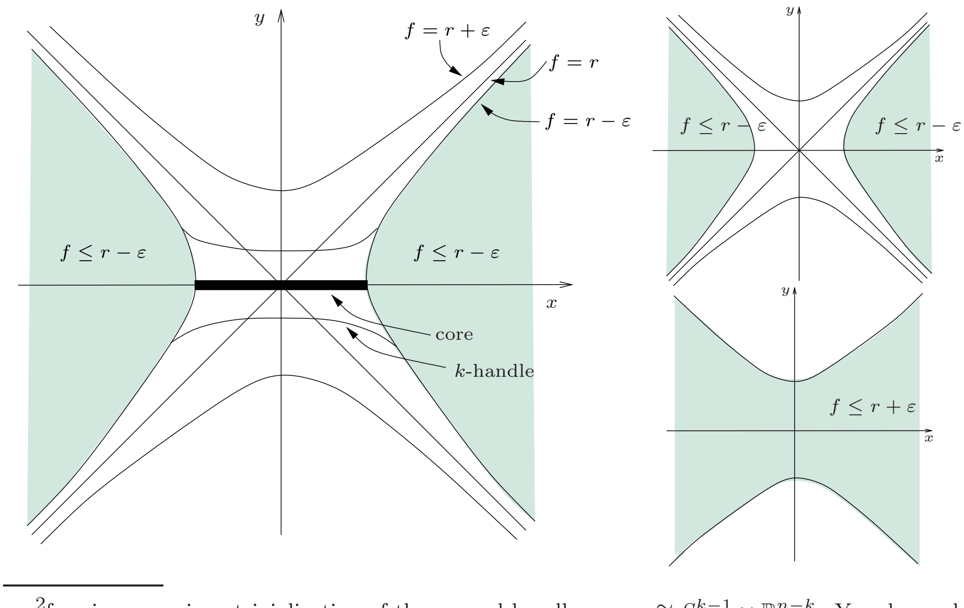
$$\begin{array}{c} f \textit{ has exactly one critical point} \\ p \in f^{-1}[a,b] \textit{ of index } k \end{array} \Rightarrow M_b \simeq M_a \cup k\textit{-cell (hpy equivalence)}$$

Rmk. *If there are several critical points on the same level set, then the proof still works (attach several cells).*

Proof. Let $r = f(p)$. By Theorem 3.5, we just need to show

$$M_{r+\varepsilon} \simeq M_{r-\varepsilon} \cup k\text{-cell} \qquad (\text{for small } \varepsilon > 0)$$

Pick Morse chart near p: $\boxed{f(x,y) = r - |x|^2 + |y|^2,\ (x,y) \in \mathbb{R}^k \times \mathbb{R}^{m-k}}$.

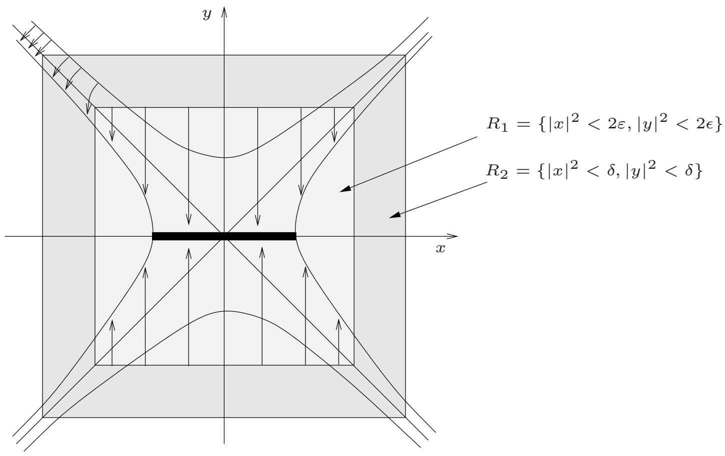


[2]framing = a given trivialization of the normal bundle $\nu_{\mathrm{im}(\varphi)} \cong S^{k-1} \times \mathbb{R}^{n-k}$. You demand the normal bundle to be trivial, since you want to extend the knot embedding to an embedding $S_k = S^{k-1} \times D^{n-k} \hookrightarrow \partial X$.

Aim: $M_{r+\varepsilon}$ deformation retracts onto $M_{r-\varepsilon} \cup$ core, where

$$\begin{aligned} \text{core} &= \{(x,0) : f(x,0) = r - |x|^2 \geq r - \varepsilon\} \\ &= \{(x,0) : |x|^2 \leq \varepsilon\} = k\text{-cell.} \end{aligned}$$

We will define two regions, R_1, R_2:



Aim: the deformation is the flow of a vector field v which is $\left\{ \begin{smallmatrix} -\nabla f = (2x, -2y) \text{ outside } R_2 \\ \text{vertical } = (0, -2y) \text{ in } R_1 \end{smallmatrix} \right.$
Pick $0 < \delta < 1$ so that $R_2 \subset$ Morse chart.
Pick a Riemannian metric on M which equals[3] the Euclidean metric on R_2.
Pick $\varepsilon < \delta/2$. Define

$$\boxed{v = \left\{ \begin{smallmatrix} (2\beta(x,y) \cdot x, -2y) \text{ on } R_2 \\ -\nabla f \text{ outside } R_2 \end{smallmatrix} \right.}$$

with $\beta : \mathbb{R}^m \to [0,1]$ equal to 0 on R_1 and 1 outside R_2. Let

$$\begin{aligned} \varphi(t,z) &= \text{flow of } \tfrac{v}{|v|} \text{ for time } t \text{ starting from } z \in M_{r+\varepsilon}, \text{ where} \\ & \quad 0 \leq t \leq T(z) = \text{ time required to reach } M_{r-\varepsilon} \cup \text{ core.} \end{aligned}$$

Note: if $z \in M_{r-\varepsilon}$ then $T(z) = 0, \varphi(0,z) = z$. Moreover, $T(z)$ is finite since:[4]

$$df \cdot \frac{v}{|v|} = \left\{ \begin{array}{l} -\|\nabla f\| \text{ outside } R_2 \\ (-2xdx + 2ydy)(2\beta x \partial_x - 2y\partial_y) = -4\beta|x|^2 - 4|y|^2 \leq -4|y|^2 \text{ in } R_2 \end{array} \right.$$

(so you can't stop flowing before reaching $M_{r-\varepsilon}$ or $y = 0$).

Finally, observe that $T(z)$ is continuous. Conclusion:
$(t,z) \mapsto \varphi(t \cdot T(z), z)$, for $t \in [0,1]$, deform retracts $M_{r+\varepsilon}$ onto $M_{r-\varepsilon} \cup k$-cell. $\square$

Rmk. *The proof also shows:*

- M_b, M_r *deform retract onto* $M_a \cup k$*-cell*
- $M_b \cong M_a \cup k$*-handle are diffeomorphic*
- $f^{-1}(b) =$ *surgery on* $f^{-1}(a)$.

Def. *Recall a* CW structure *on* M *are subsets* $M_0 \subset M_1 \subset \ldots \subset M_n = M$ *where* M_0 *is a discrete set of points, and*

$$M_k = M_{k-1} \cup D_1^k \cup D_2^k \cup \cdots$$

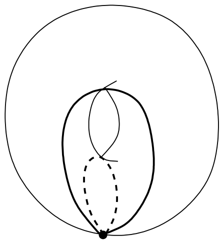
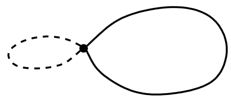
[3]recall metrics can be patched together by a partition of unity argument.
[4]abbreviating $xdx = \sum x_i dx_i$, etc.

is obtained by attaching k-cells via continuous maps $f_i^k : \partial D_i^k \to M_{k-1}$.

Cor. *M closed, $f : M \to \mathbb{R}$ Morse. Then M has a CW structure with a k-cell attached for each critical point of f of index k, and $M_0 =$ minima of f.*

Proof. For a *self-indexing Morse function*[5] the result is immediate from the Theorem (consider M_n by induction on n). So the result follows by Hwk 9: one can modify a Morse f to make it self-indexing without changing $\mathrm{Crit}(f)$ and its indices.

Alternative: use hpy theory (cellular approximation theorem[6]) to overcome the issue that the Thm is not attaching cells ordered by dimension.[7] □

Torus =  =  ∪ 2-cell

= a figure eight with a 2-cell attached.

Rmk. *Provided $M_r = (f \le r)$ is compact, the Thm holds for non-compact M (for the Corollary you need a little care using homotopy theory[7]).*

3.8. **Stable and unstable manifolds.** Let φ be the flow of a vector field v on M.

Def. *For $v(p) = 0$,*

$$\begin{aligned} \textit{unstable mfd} \quad W^u(p,v) &= \{x \in M : \lim_{t\to-\infty} \varphi(x,t) = p\} \\ \textit{stable mfd} \quad W^s(p,v) &= \{x \in M : \lim_{t\to+\infty} \varphi(x,t) = p\} \end{aligned}$$

Rmk. *This of these as the points flowing "out of"/"into" p. They both contain p since $\varphi(p,t) = p$. For nasty v the W^u, W^s may not be manifolds.*

Def. *For $v = -\nabla f$ we abbreviate $W^u(p) = W^u(p, -\nabla f), W^s(p) = W^s(p, -\nabla f)$, which are often called* descending mfd $\mathcal{U}(p)$ *and* ascending mfd $\mathcal{A}(p)$ *(thinking of f as a "height" function).*

Example. In the pf of Thm 3.7, $-\nabla f = (-2x, 2y)$ near p (Euclidean metric) so

$$W^u(p) = x\text{-plane} \qquad W^s(p) = y\text{-plane} \qquad (\text{near } p)$$

so $\boxed{M_{r+\varepsilon} = M_{r-\varepsilon} \cup W^u(p)}$ and infinitesimally: $\boxed{T_pM = T_pW^u(p) \oplus T_pW^s(p)}$ $(*)$

Observe $W^u(p)$ is a mfd in a nbhd U of p, so it is globally a mfd since

$$W^u(p) = \bigcup_{t \ge 0} \varphi(W^u(p) \cap U, t),$$

so $\boxed{W^u(p) \cong \mathbb{R}^{\mathrm{index}(p)}}$ (exercise). Similarly, $W^s(p)$ is[8] a submfd $\mathbb{R}^{m-\mathrm{index}(p)} \hookrightarrow M$.

[5] meaning the index $|p| = f(p)$.

[6] Maps $S^n \to M$ can be homotoped to maps $S^n \to M_n \subset M$, which is an immediate consequence of Sard's theorem: first smoothen the map, then by dimensions the map cannot surject onto the higher discs D_i^m for $m > n$, so you can homotope it to avoid those discs.

[7] For details, see Milnor's *Morse theory* book.

[8] $W^u(p), W^s(p)$ are embedded submfds, but the embedding is not proper.

Thm (Hadamard-Perron & Hartman-Grobman)**.**
$W^u(p,v), W^s(p,v)$ *are "mfds" for* $\begin{cases} v = -\nabla f & (f \textit{ Morse, for any metric}) \\ v = \textit{vector field with hyperbolic fixed points}^9 \end{cases}$
and $(*)$ *holds.*

More precisely, respectively $\begin{cases} W^u(p,-\nabla f), W^s(p,-\nabla f) \textit{ are embedded planes,} \\ W^u(p,v), W^s(p,v) \textit{ are injectively immersed planes.} \end{cases}$

Rmk. *For general hyperbolic v a point may return arbitrarily close to itself under the flow: for example, for v you may have non-constant v-flowlines converging to p at both ends. This does not happen for $-\nabla f$ because f decreases along the flow.* Non-examinable: prove that $-\nabla f$ has hyperbolic fixed points, so the first case is a particularly well-behaved special case of the second.

Rmk. *The proof is hard. If you are* given *a metric, then you cannot hope that in a Morse chart the metric is Euclidean (e.g. consider a metric which is not flat near p). Therefore you do not know what $-\nabla f$ looks like near the critical points, so even the infinitesimal result (do $TW^u(p), TW^s(p)$ exist?) is hard.*

Two tricks to avoid the Theorem.

(1) change the metric to make it Euclidean near p via a hpy
$$g_t = (1-t)\cdot g + t\cdot((1-\beta)\cdot g + \beta \cdot g_{\text{Euclidean}})$$
using a bump function $\beta = 1$ near p (compare pf of Thm 3.7).

(2) don't use ∇f, and instead use a *gradient-like vector field* v (see Hwk 7):
$v(f) > 0$ except at $\operatorname{Crit} f$, $v = \nabla_{\text{Euclidean}} f$ in a Morse chart near $p \in \operatorname{Crit} f$.

[9] meaning, if $v(p) = 0$, then $d\varphi(p,t)$ has no evalues λ with $|\lambda| = 1$ $\forall t \neq 0$.