LECTURE 9.

PART III, MORSE HOMOLOGY, 2011

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Motivation. Denote the (parametrized) $-\nabla f$ flowlines from p to q by:

$$\boxed{W(p,q) = \{u : \mathbb{R} \to M : u' = -\nabla f, \lim_{s\to-\infty} u(s) = p, \lim_{s\to+\infty} u(s) = q\}}$$

Identify $W(p,q) \equiv W^u(p) \cap W^s(q), u \leftrightarrow u(0)$. When is $W(p,q)$ a manifold?

3.9. **Classical approach.** $f : M \to \mathbb{R}$ Morse, and g is a Riem metric on M.

(1) Thm 3.8: Prove $W^u(p)$, $W^s(q)$ are smooth mfds of dim $= |p|, m - |q|$.
(2) ***Def.*** *$f : M \to \mathbb{R}$ is* Morse-Smale *for the Riem metric g on M if*

$$W^u(p) \pitchfork W^s(q) \quad \forall p, q \in \mathrm{Crit}(f)$$

By Cor 1.4, (f,g) Morse-Smale $\Rightarrow W(p,q) \equiv W^u(p) \cap W^s(q)$ is mfd of codim $= (m - |p|) + |q|$, hence of dimension:

$$\dim W(p,q) = |p| - |q|.$$

Rmk. $\Rightarrow W(p,p) = \{\text{constant } p\}$ since f decreases along the flow.
Rmk. $\Rightarrow W(p,q) = \emptyset$ if $|p| \leq |q|$ $(p \neq q)$ since a non-constant flowline u gives a 1-dim family of flowlines $u(\cdot + \text{constant})$.

(3) **Thm [Kupka-Smale 1963]**
(M,g) closed Riem mfd $\Rightarrow$ generic $f \in C^\infty(M,\mathbb{R})$ are Morse-Smale.

Example. Hwk 1: 1(ii) Morse-Smale, but 1(i) is not by Rmk ($\exists$ 2 flowlines between the index 1 crit pts)

Rmk. The Kupka-Smale Thm is stronger: for generic smooth vector fields, the fixed pts are hyperbolic[1] and $W^u(p) \pitchfork W^s(q)$.

(4) One can describe how the $W^u(p)$'s fit together, so get a CW structure on M and can do cellular homology.

3.10. **Cellular homology for self-indexing f.**

Thm (Thom, Smale, Milnor $\sim$ 1965)**.** *For f self-indexing, and g Morse-Smale, $H_*(M) \cong$ homology of* $\bigoplus_{p\in Crit(f)} \mathbb{Z}\,p$ *with differential*

$$d\,p = \sum_{|q|=|p|-1} (S^u(p) \cdot S^s(q))\,q$$

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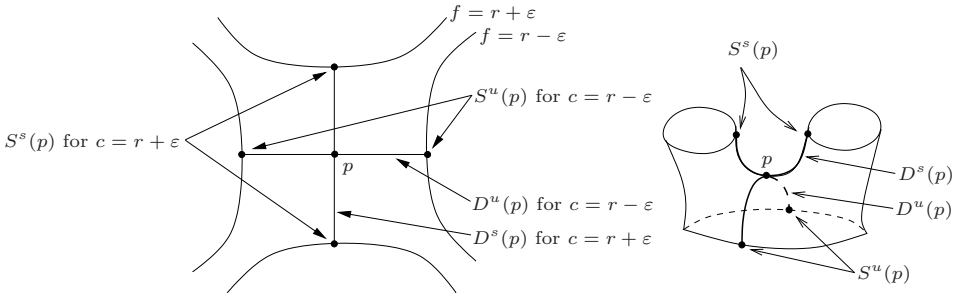
[1]so W^u, W^s are injectively immersed submfds, see 3.8.

where the intersection number $S^u(p) \cdot S^s(q)$ is calculated[2] *inside $f^{-1}(c)$, where you fix any regular value $c \in (f(q), f(p))$, and where:*[3]

$$\begin{array}{lcll} S^u(p) & = & W^u(p) \cap f^{-1}(c) & \textit{unstable sphere} \\ D^u(p) & = & W^u(p) \cap (f \geq c) & \textit{unstable disc bounded by } S^u(p) \end{array}$$



For f self-indexing Morse:

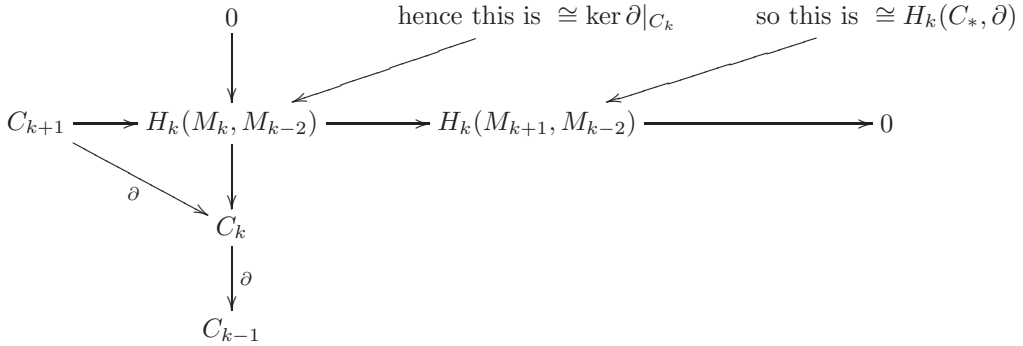
$$\begin{array}{rcl} C_k & = & H_k(M_k, M_{k-1}) \\ & \cong & H_k\left(M_k/M_{k-1} = \text{bouquet of } k\text{-spheres one for each } p \in \mathrm{Crit}(f), |p| = k\right) \\ & \cong & \bigoplus_{|p|=k} \mathbb{Z}\, p \end{array}$$

Fact.[4] $S \subset M \subset L \Rightarrow \mathrm{LES}^5$ $H_*(M,S) \xrightarrow{\text{incl}} H_*(L,S) \xrightarrow{\text{quot}} H_*(L,M) \xrightarrow{\partial} H_{*-1}(M,S)$.
Define. $\partial : C_k \to C_{k-1}$ from LES of $M_{k-1} \subset M_k \subset M_{k+1}$.

Recall this is how cellular homology is defined (and one checks that $\partial^2 = 0$).

Thm. $H_*(M) \cong H_*(C_*, \partial)$

Proof. Use the 2 LES of triples:



$\Rightarrow H_k(C_*, \partial) \cong H_k(M_{k+1}, M_{k-2}) \cong H_k(M_{k+1}) \cong H_k(M)$. □

[2]The point is that you can't do $W^u(p) \cdot W^s(q)$ because it has dimension ≥ 1 because of the reparametrization freedom $u(\cdot + \text{constant})$.

[3]these are a sphere/disc if you choose c very close to $f(p)$, using the local model of Thm 3.7, and hence it's true for any $c \in (f(q), f(p))$ by using the flow of $-\nabla f$ (see Thm 3.5).

[4]Small, Medium, Large.

[5]Long Exact Sequence.

Intersection Pairing.
$A, B \subset X$ closed mfds, $\operatorname{codim}(A) = k$, $\dim B = k$, assume $A \pitchfork B = \{p_1, p_2, \ldots\}$ and assume A is co-oriented (the normal bundle ν_A is oriented[6])

Recall the **Thom isomorphism**:

$$\begin{array}{rcl} H_0(A) & \cong & H_k(X, X \setminus A) \\ [\text{pt}] & \mapsto & \tau = \text{Thom class} = \text{fibrewise an orienation generator of } \nu_A \end{array}$$

Idea: it's a family of generators of $H_k(D^k, \partial D^k)$ where D^k is the disc in the fibre.

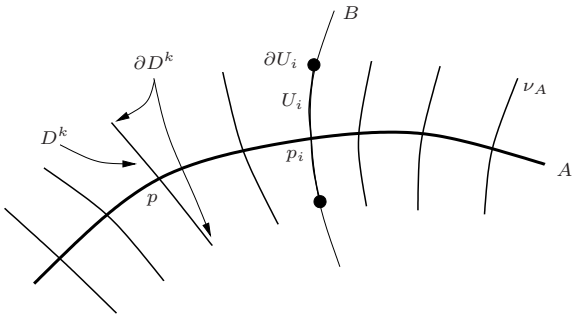
Thm 1.

$$\begin{array}{ccccc} H_k(B) & \xrightarrow{\text{incl}} & H_k(X) & \xrightarrow{\text{quot}} & H_k(X, X \setminus A) \\ [B] & \mapsto & [B] & \mapsto & (A \cdot B)\,\tau. \end{array}$$

Proof. Localize problem: pick disjoint open balls U_i around p_i in B. By naturality of the Thom iso

$$\begin{array}{ccc} H_k(U_i, U_i - p_i) & \overset{\text{incl}}{\cong} & H_k(X, X \setminus A) \\ [U_i, U_i - p_i] & \mapsto & \pm\tau \end{array}$$

where $\pm$ is the intersection number of A, B at p_i (since you can check the intersection number is the comparison of the generator $[D^k, \partial D^k]$ for $(\nu_A)_{p_i}$ with $[U_i, \partial U_i]$).



Now globalize:

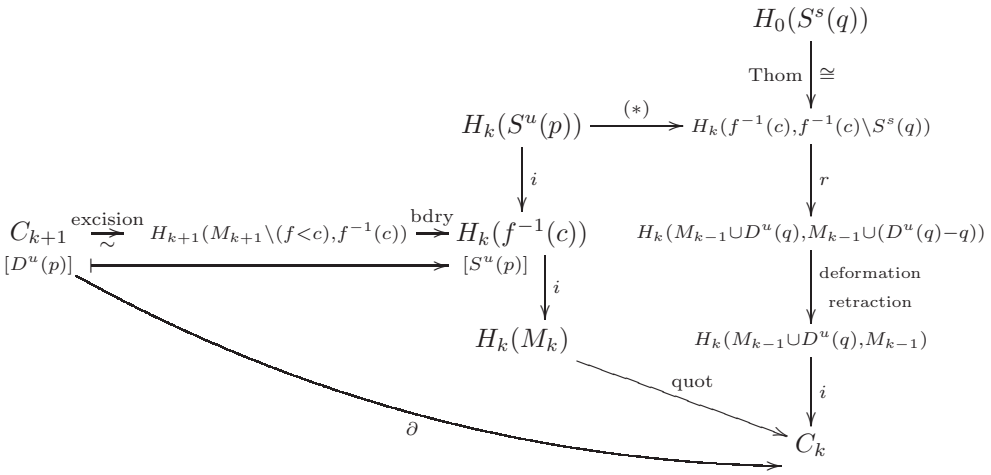
$$\begin{array}{ccccc} H_k(B) & \longrightarrow & H_k(X) & \longrightarrow & H_k(X, X \setminus A) \\ & \searrow & & & \uparrow \\ & & H_k(B, B \setminus A \cap B) & \xrightarrow[\text{excision}]{\cong} & \bigoplus_i H_k(U_i, U_i - p_i) \end{array}$$

□

Thm 2. $C_{k+1} \to C_k$, $p_i \mapsto \sum_j (S^s(q_j) \cdot S^u(p_i))\,[D^u(q_j)]$
(intersecting inside $f^{-1}(c)$, where one can take $c = k + \frac{1}{2}$).

[6] you can ignore orientations if you count intersections modulo 2.

Proof. For simplicity, do the case where C_{k+1} is generated by just p, $|p| = k+1$ and C_k by just q, $|q| = k$ (the proof easily generalizes).



where i stands for "inclusion", r is the (restriction of the) retraction of M_k onto $M_{k-1}\cup D^u(q)$ of Thm 3.7, and the arrow "deformation retraction" pushes $D^u(q)-q$ away from q and into M_{k-1}.

Finally, one can check that this diagram commutes.[7]

By Thm 1, the arrow $(*)$ is

$$[S^u(p)] \mapsto (S^s(q) \cdot S^u(p))\,\tau$$

and the right column is $[\text{pt}] \mapsto \tau \mapsto [D^u(q)] \in C_k$ (easy exercise). □

3.11. **Modern vs classical.**

Def (Witten (1982))**.** *The* Morse complex *is* $MC_* = \bigoplus_{p\in Crit(f)} \mathbb{Z}\,p$. *For* $p \neq q \in Crit(f)$ *the* moduli space *of Morse trajectories*[8] *is*

$$\boxed{\mathcal{M}(p,q) = W(p,q)/\mathbb{R}}$$

where the $\mathbb{R}$ *action by time-shifting identifies* $u \sim u(\cdot + constant)$.

The Morse differential *is*

$$\boxed{\partial p = \sum_{|q|=|p|-1} \#\mathcal{M}(p,q)\cdot q}$$

where $\#\mathcal{M}(p,q)$ *counts the number of elements with orientation signs* (*exercise: this agrees with* $S^u(q)\cdot S^s(p)$).

Cor. *For a self-indexing Morse function* f *and a Morse-Smale metric* g *on a closed manifold* M*, the Morse differential satisfies* $d^2 = 0$ *and the Morse homology*[9] *is isomorphic to ordinary homology*

$$MH_*(f) \cong H_*(M).$$

[7] the two big columns commute at the level of topological maps if you replace $i : f^{-1}(c) \to M_k$ by r, and then since $r_* = i_*$ on homology it commutes also at the level of homology.

[8] called *instantons* or *tunneling paths* by physicists.

[9] $MH_*(f,g) = H_*(MC_*, d)$.