LECTURE 10.

PART III, MORSE HOMOLOGY, 2011

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3.11. Modern vs classical (continued).

$$\begin{tikzcd}
(f,g) \arrow[r, Rightarrow, "\text{perturb } f"] & W^u(p) \pitchfork W^s(q) \arrow[r, Rightarrow] & W(p,q) \text{ mfd} \arrow[r, Rightarrow] & \mathcal{M}(p,q) = W(p,q)/\mathbb{R} \text{ mfd} \\
f \text{ Morse} \arrow[rrr, Rightarrow, bend right, "\text{modern approach}", "\text{perturb } g"'] & \forall p \neq q \in \mathrm{Crit}(f) & \dim = |p| - |q| & \dim = |p| - |q| - 1
\end{tikzcd}$$

Rmk. *From a dynamical systems perspective, the natural object to perturb is the dynamical system: f. You would not "shake the universe" to get the equations that you want. The modern approach is however reasonable: you can never measure exactly what the Riemannian metric g is, so you can always assume it is generic.*

Historical context:

1988 Floer:	$\mathcal{M}(p,q)$ = zero set of a Fredholm section of a Banach vector bundle and proved smoothness & compactness theorems
1989 Floer:	proved (an infinite dimensional generalization of) $\partial^2 = 0$, and $MH_*(f) \cong H_*(M)$ using the "Conley index"[1]
1990 Salamon:	proved $MH_*(f) \cong H_*(M)$ using the Conley index.

Importance of the modern approach:

Need to avoid 3.9 (i): in infinite dimensional generalizations of Morse theory,

(1) $W^u(p), W^s(q)$ are usually not defined[2]
(2) and even when they are defined, they are usually infinite dimensional.[3]

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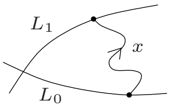
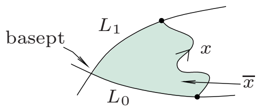
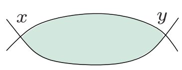
[1] which is essentially a sophisticated dynamical version of cellular homology.

[2] because flows are badly behaved in infinite dimensions.

[3] So you have a bad intersection theory. Example: take the space of sequences of real numbers. You can shift sequences by a fixed amount without losing information, so you can arbitrarily increase codimensions of subsets, so can deform any two proper subsets so that they never intersect.

3.12. **Modern generalizations.** [Non-examinable]

Homology	Manifold	Morse function f	$x \in \mathrm{Crit}(f)$	$-\nabla f$ flowline in $\mathcal{M}(x,y)$
Morse (MH)	M^n closed	$f : M \to \mathbb{R}$	$x \in M : d_x f = 0$	$u : \mathbb{R} \to M$ $\partial_s u = -\nabla f$
	Applications: count # of critical points of a function.			
∞-dim MH	$\mathcal{L}N = C^\infty(S^1, N)$ free loop space	$\mathcal{E} : \mathcal{L}N \to \mathbb{R}$ $\mathcal{E}(x) = \int_0^1 \frac{1}{2}\lvert x'\rvert^2\, dt$	$x =$ closed geodesic of period $= 1$	$u : \mathbb{R} \times S^1 \to M$ $\partial_s u = -\nabla \mathcal{E}$
	Applications: count # of closed geodesics.			
Symplectic (SH)	$\mathcal{L}(T^*M)$ T^*M coords (q,p), $\omega = d(pdq)$ $= dp \wedge dq$ symplectic (closed 2-form ω s.t. $\omega^n = \mathrm{vol}_{T^*M}$)	$A_H(x) = -\int_x p\, dq + \int_x \frac{1}{2}\lvert p\rvert^2$ (if x contractible, it equals "area" $\int_{\overline{x}} \omega$ of disc $\overline{x}$ bounding x (Stokes))	$x =$ closed geodesic of period $= 1$ $T^*M \cong TM \ni (x, x')$	Floer's eqn $u : \mathbb{R} \times S^1 \to T^*M$ $\partial_s u = -J\partial_t u - \nabla \frac{1}{2}\lvert p\rvert^2$ (compare Cauchy-Riemann eqns: $\partial_s u = -i\partial_t u$ on $\mathbb{C}$)
	Applications: count # of closed geodesics.			
Floer (FH)	$\mathcal{L}_0 N =$ contractible loops (N^{2n}, ω) symplectic closed mfd[a]	$A_H : \mathcal{L}_0 N \to \mathbb{R}$ $A_H(x) = -\int_{\overline{x}} \omega + \int_x H$	Hamiltonian orbits $x' = X_H(x)$ $(\omega(\cdot, X_H) = dH)$	Floer's eqn $u : \mathbb{R} \times S^1 \to M$ $\partial_s u = -J\partial_t u - \nabla H$
	Applications: count # of contractible Hamiltonian orbits.			
Lagrangian FH	$\Omega(L_0, L_1) =$ paths in (N^{2n}, ω) from L_0 to L_1 (Lagrangian submfds:[b] $\omega\vert_L = 0, \dim L = n$) 	$A(x) = -\int_{\overline{x}} \omega$ 	$x \in L_0 \cap L_1$	"holomorphic" strips $u : \mathbb{R} \times S^1 \to N$ $\partial_s u = -J\partial_t u$ 
	Applications: package up LFH's $\forall$Lagrangians and all algebraic operations $\Rightarrow$Fukaya category (Fukaya,Seidel,...) $\overset{\text{dictionary}}{\longleftrightarrow}$ *Category of coherent sheaves on a mirror manifold* *SYMPLECTIC TOPOLOGY* $\overset{\text{mirror symmetry}}{\longleftrightarrow}$ *ALGEBRAIC GEOMETRY*			
	Applications: Heegaard Floer homology "=" LFH[c] for 3-mfds (Ozsváth-Szabo,...) *$\Rightarrow$ invariants of knots (detects knot genus, categorifies Alexander polynomial)*			
Instanton (IH)	$B = \{$connections x on $P\}/\sim$ ($\sim$gauge eq., $P \to Y$ principal $SU(2)$-blde, Y closed 3-mfd)	Chern-Simons functional $CS(x) = \frac{1}{8\pi^2}\int_Y \mathrm{Tr}(x \wedge dx +$ $+ \frac{2}{3} x \wedge x \wedge x)$	flat connections x	$u : \mathbb{R} \to B$ $\partial_s u = - * F_u$ ($*$ =Hodge star, F_u =curvature of $u(s)$)
Seiberg- -Witten (HM)	$\{$spin^c-connections x with section $Y \to S$ to a reference spin bdle$\}$	Chern-Simons-Dirac functional[d]	monopoles x	4-dim SW eqns
	Applications: invariants of 4-mfds (a 4-dim coborism gives a hom of HM^*'s of boundary 3-mfds) *Taubes (2010): Seiberg-Witten homology $\cong$ Heegaard Floer homology*			

[a]Meaning, ω is a closed 2-form with $\omega^n = \mathrm{vol}_M$. Usually you assume $\int_{\text{2-spheres}} \omega = 0$ so that $A_H : \mathcal{L}_0 N \to \mathbb{R}$ is well-defined independently of the choice of disc $\overline{x}$ bounding x. This $\omega[\pi_2(M)] = 0$ condition also prevents an issue called "bubbling": non-constant holomorphic spheres don't appear because they would have positive ω-area.

[b]Called (Lagrangian) *branes* by physicists.

[c]Loosely speaking, you mimick the construction of LFH for $N =$ the symmetric product $\mathrm{Sym}^g(\Sigma_g)$ of the surface Σ_g arising in a Heegaard splitting of the 3-mfd, with $L_0 = \prod \alpha_i$, $L_1 = \prod \beta_i$ for appropriate loops α_i, β_i which generate the H_1 of Σ_g.

[d]See Kronheimer and Mrowka, *Monopoles and Three Manifolds.*

3.13. **Key aspects of modern homology theories.**

(1) ***Moduli Space:*** $\mathcal{M}(x,y)$ = zero set of some Fredholm map;
(2) ***Transversality:*** perturb parameters $\Rightarrow \mathcal{M}(x,y)$ smooth mfd;
(3) ***Dimension:*** index of some Fredholm operator[4] in terms of x, y gradings;
(4) ***Compactness & Gluing:*** $\mathcal{M}(x,y)$ compact if allow "broken solutions";
(5) ***Orientations:*** for $\mathbb{Z}$ coefficients, need to choose[5] orientations for $\mathcal{M}(x,y)$;
(6) ***Invariance:*** $H_*(\oplus_{x\in \mathrm{Crit}(f)}\mathbb{Z}\,x, \partial$ counting $\mathcal{M}(x,y)$'s) indep of parameters.

4. Transversality Theorem

4.0. **Transversality for $\mathcal{M}(p,q)$: outline of the proof.**
For $f : M \to \mathbb{R}$ smooth Morse function, consider

$$\begin{array}{c} E = \text{all vector fields along paths} \\ \big\downarrow \;\; \big) F(u,g) = \partial_s u - \nabla^g f \\ U \times G = \{\text{all paths } \mathbb{R} \to M \text{ from } p \text{ to } q\} \times \{\text{all metrics}\} \end{array}$$

where $\nabla^g f$ is ∇f for g: $g(\nabla^g f, \cdot) = df$. Then notice:

$$\boxed{\mathcal{M}^g(p,q) = (\text{ zero set of } F_g = F(\cdot, g))}$$

Aim: Parametric transversality $\Rightarrow \mathcal{M}^g(p,q)$ is smooth mfd for generic g.

What is needed: U, G, E Banach mfds, $F \pitchfork 0_E$, DF_g Fredholm.

What Banach mfd structure?

- Cannot use C^∞: not Banach
- $\boxed{G = \{C^k\text{-metrics on } M\}}$ ✓
- $\partial_s u = -\nabla^g f$ and $u \in C^1 \Rightarrow u$ is C^{k+1}:
$$(\partial_s)^{k+1} u = (\partial_s)^k(-\nabla^g f).$$
- C^1 is not so good on non-compact domains like $\mathbb{R}$. Also: want to integrate $\int_{\mathbb{R}} |u'|^2\,dt$ and $\int_{\mathbb{R}} |\nabla f_u|^2\,dt$, but $C^1 \cap L^2$ is not Banach. Try $W^{1,2}$ instead.
- $W^{1,2}$ is great since Hilbert: for smooth $u, v : \mathbb{R} \to M \subset \mathbb{R}^a$, inner product[6]
$$\langle u, v\rangle_{1,2} = \langle u, v\rangle_{L^2} + \langle \partial_s u, \partial_s v\rangle_{L^2}.$$
Recall $W^{1,2}$ is obtained by completing $C^1 \cap L^2$ for the induced norm $\|\cdot\|_{1,2}$. Hilbert spaces are great because you can tell if an operator is surjective by studying the subspace perpendicular to the image.
- First attempt:
$$\boxed{U = \{u \in W^{1,2}(\mathbb{R}, M) : u \to p, q \text{ at the ends}\}} \; \textit{(not quite correct)}$$
$$\boxed{E = \{L^2 - \text{vector fields over paths}\}}$$

However, various problems arise:

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[4] the linearization of the Fredholm map in (1).
[5] chosen compatibly with respect to the gluing (4) of $\partial\mathcal{M}(x,y)$ onto $\mathcal{M}(x,y)$.
[6] $\langle u, v\rangle_{L^2} = \int u(s)\cdot v(s)\,ds$ using the dot product in $\mathbb{R}^a$ after embedding $M \hookrightarrow \mathbb{R}^a$.

- Does $u \to p, q$ even make sense? For smooth $u \in W^{1,2}$ get
$$\begin{aligned} |u(b) - u(a)| &= |\int_a^b (\partial_s u)\, ds| \\ &\leq \int_a^b |\partial_s u|\, ds \\ &\leq (\int_a^b 1^2)^{1/2} (\int_a^b |\partial_s u|^2)^{1/2} \\ &\leq \sqrt{b-a} \cdot \|u\|_{1,2} \end{aligned}$$
does this mean that elements in $W^{1,2}$ can be represented by a continuous function? (in which case $u \to p, q$ makes sense).
- even if u is continuous, F involves "$\partial_s u$": surely u is not C^1 in general?
- finally, $W^{1,2}(\mathbb{R}, M)$ doesn't quite make sense: a constant $\mathbb{R} \to \{x\} \in M \subset \mathbb{R}^a$ has norm $\int_{\mathbb{R}} |x|\, ds = \infty$ (for $x \neq 0$), and most other maps $\mathbb{R} \to M$ also have infinite norm. So in fact we want u to be *locally* $W^{1,2}$, and we require that u is $W^{1,2}$ at the ends with norms calculated in such a way that the critical points have norm zero. This requires some care.

Conclusion: we need to review the Sobolev spaces $W^{1,2}$ before moving on. Then we will define G, U, E properly, and prove they are Banach mfds. Finally we will prove the hypotheses required for parametric transversality. The upshot will be that the moduli spaces $\mathcal{M}(p, q)$ are smooth mfds for generic metrics g.