LECTURE 11.

PART III, MORSE HOMOLOGY, 2011

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SOBOLEV SPACES

The book by Adams, *Sobolev spaces*, gives a thorough treatment of this material. We will treat Sobolev spaces with greater generality than necessary (we only use $W^{1,2}$ and L^2), since these spaces are ubiquitously used in geometry.

4.1. **$W^{k,p}$ spaces on Euclidean space.** Notation: $k \geq 0$ integer, $1 \leq p < \infty$ real.

Def. *For an open set $X \subset \mathbb{R}^n$, $W^{k,p}(X) = W^{k,p}(X, \mathbb{R})$ is the completion*[1] *of $C^\infty(X) = \{smooth\ u : X \to \mathbb{R}\}$ with respect to the $\|\cdot\|_{k,p}$-norm*[2]

$$\|u\|_{k,p} = \sum_{|I| \leq k} \|\partial^I u\|_p = \sum_{|I| \leq k} \left(\int_X |\partial^I u|^p \, dx \right)^{1/p}$$

$W^{k,\infty}(X)$ is defined analogously using $\|u\|_{k,\infty} = \sum_{|I|\leq k} \sup |\partial^I u|$.

Def. *$W^{k,p}(X, \mathbb{R}^m)$ is the completion of $C^\infty(X, \mathbb{R}^m)$ using*

$$\|u\|_{k,p} = \sum_{i=1,\ldots,m} \|u^i\|_{W^{k,p}(X)}$$

where u^i are the coordinates of u. An equivalent norm[3] *can be defined using the previous definition, replacing $|\partial^I u|$ by $|\partial^I u|_{\mathbb{R}^n}$.*

Rmks.

(1) *C^∞ is dense in C^k with respect to $\|\cdot\|_{k,p}$, so completing C^k is the same as completing C^∞.* ***Fact.*** *When ∂X smooth (or C^1), $C^\infty(\overline{X}) \subset W^{k,p}(X)$ is dense ($\overline{X}$ = closure of $X \subset \mathbb{R}^n$), so it is the same as completing $C^\infty(\overline{X})$.*

(2) *$W_0^{k,p}$ is the completion of C_c^∞ inside $W^{k,p}$, where*[4]

$$C_c^\infty(X) = \{smooth\ compactly\ supported\ functions\ \phi : X \to \mathbb{R}\}$$

These spaces typically arise in geometry when you globalize a locally defined function after multiplying by a bump function.
Example. *$\phi \in C_c^\infty(X)$, $u \in W^{k,p}(X) \Rightarrow \phi \cdot u \in W_0^{k,p}(X)$.*
Warning. *Usually $W_0 \neq W$ since the u's must $= 0$ on ∂X, unlike $C^\infty(\overline{X})$.*

[1] it is always understood that we complete the subset of C^∞ of u's with *bounded* $\|u\|_{k,p}$.

[2] where $I = (i_1, i_2, \ldots, i_n)$, $\partial^I = (\partial_1)^{i_1} \cdots (\partial_n)^{i_n}$, $\partial_j = \frac{\partial}{\partial x_j}$, $|I| = i_1 + \cdots + i_n$.

[3] Two norms $\|\cdot\|$, $\|\cdot\|'$ are *equivalent* if $\exists$ constants $a, b > 0$ such that $a\|x\| \leq \|x\|' \leq b\|x\|$, $\forall x$.

[4] The *support* of ϕ is $\mathrm{supp}(\phi) = \overline{\{x \in X : \phi(x) \neq 0\}}$.

(3) $W^{k,p}_{loc}$ = locally $W^{k,p}$ maps = *completion of C^∞_c with respect to the topology:*[5]

$$u_n \to u \Leftrightarrow u_n|_C \to u|_C \quad \forall C \Subset X$$

Loosely think of this as saying: the restriction to any compact is $W^{k,p}$. ***Warning.*** *This is not a normed space, but it is a complete metric space.*

(4) *All these spaces are* separable*: there is a countable dense subset, namely the polynomials with rational coefficients.*

4.2. L^p theory.

$L^p = W^{0,p}$ with $\|u\|_p = \left(\int_X |u|^p\,dx\right)^{1/p}$, and $L^\infty = W^{0,\infty}$ with $\|u\|_\infty = \sup|u|$. Recall **Hölder's inequality**[6]

$$\int_X |u\cdot v|\,dx \le \|u\|_p\|v\|_q \qquad \text{for } \tfrac{1}{p}+\tfrac{1}{q}=1.$$

Lemma. $p \ge q \Rightarrow \mathrm{vol}(X)^{-1/q}\cdot\|u\|_q \le \mathrm{vol}(X)^{-1/p}\cdot\|u\|_p$
$\Rightarrow L^p(X) \hookrightarrow L^q(X)$ *is bounded for X bounded.*

Proof. For $u \in C^\infty$, let $A = \int |u|^p$, then

$$\frac{\|u\|_q}{\|u\|_p} = \frac{(\int |u|^q)^{1/q}}{A^{\frac{1}{p}}} = \left(\int \left(\frac{|u|^p}{A}\right)^{q/p}\cdot 1\right)^{1/q} \le \left[\left(\int \frac{|u|^p}{A}\right)^{q/p}\cdot\left(\textstyle\int 1\right)^{1-\frac{q}{p}}\right]^{1/q} = \mathrm{vol}(X)^{\frac{1}{q}-\frac{1}{p}}$$

using Hölder in the inequality. Therefore $(C^\infty\cap L^p, \|\cdot\|_p) \to (C^\infty \cap L^q, \|\cdot\|_q)$ is a bdd inclusion, so we can complete it:[7] $L^p \to L^q$, $[u_n]\mapsto[u_n]$. $\square$

Example. $L^\infty(X) \subset L^p(X)$ is clearly true for bdd X, and clearly false for $X = \mathbb{R}$.

Cor. $p \ge q \Rightarrow W^{k,p}(X) \hookrightarrow W^{k,q}(X)$ *is bdd for X bdd.*

Motivation. $k > k' \Rightarrow W^{k,p} \hookrightarrow W^{k',p}$ is clearly bounded, and one might even suspect that it is compact because of a mean value thm argument. So can one combine this with the Corollary and get optimal conditions on k, p simultaneously?

Def. Recall a linear map $L : X \to Y$ is *bounded* if $\|Lx\| \le c\|x\|\ \forall x$, and *compact* if any bounded sequence gets mapped to a sequence having a cgt subsequence.[8]

4.3. Sobolev embedding theorems. Let[9]

$$\boxed{p^* = \begin{cases} \frac{np}{n-kp} & \text{if } kp < n \\ \infty & \text{if } kp \ge n\end{cases}}$$

From now on, assume $X \subset \mathbb{R}^n$ open, ∂X smooth (or C^1).

Thm. $\boxed{W^{k,p}(X) \overset{bdd}{\hookrightarrow} L^q(X) \text{ for } p \le q \le p^*}$ *(require $q \ne \infty$ if $kp = n$).*

Rmk. *For X bdd one can omit $p \le q$ by the Lemma.*

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[5] recall $C \Subset X$ means C, X open and $C \subset \overline{C} \subset X$.

[6] The generalization of the Cauchy-Schwarz inequality ($p = q = 2$).

[7] the inequality shows that L^p-Cauchy implies L^q-Cauchy, and the map $[u_n]\mapsto[u_n]$ is well-defined since $u_n \to 0$ in L^p implies $u_n \to 0$ in L^q, again by the inequality.

[8] equivalently: the closure of the image of the unit ball is compact.

[9] Unexpected results happen at the Sobolev borderline $kp = n$. Example: $\log\log(1+\frac{1}{|x|})$ on the unit ball in $\mathbb{R}^n$ is $W^{1,n}$ but neither C^0 nor L^∞.

Cor. *Under the same assumptions, $W^{k+j,p}(X) \hookrightarrow W^{j,q}(X)$ is bdd.*

Proof. Idea: $u \in W^{j,q} \Leftrightarrow \partial^I u \in L^q$, $\forall |I| \leq j$. For smooth u (afterwards complete):

$$\|u\|_{j,q} = \sum_{|I|\leq j} \|\partial^I u\|_q \leq c \sum_{|I|\leq j} \|\partial^I u\|_{k,p} \leq c' \, \|u\|_{k+j,p}. \qquad \square$$

Thm.[10] $\boxed{W^{k+j,p}(X) \overset{bdd}{\hookrightarrow} C^j_b(\overline{X}) \text{ for } kp > n}$

Thm (Rellich)**.** *X bdd & inequalities are strict $\Rightarrow$ above embeddings are compact.*

Example. $W^{1,2}(\mathbb{R}, \mathbb{R}^m) \hookrightarrow C^0_b(\mathbb{R}, \mathbb{R}^m) = \{\text{bdd cts } \mathbb{R} \to \mathbb{R}^m\}$ ($kp = 2 > n = 1$✓).
$W^{1,2}(\mathbb{R}, \mathbb{R}^m) \xrightarrow{\text{restr}} W^{1,2}((0,1), \mathbb{R}^m) \hookrightarrow C^0([0,1], \mathbb{R}^m)$ is compact.

Idea of Proof of First Theorem for $kp < n$, X bdd
Note: $kp < n$, $q \leq p^* \Leftrightarrow 0 > k - \frac{n}{p} \geq -\frac{n}{q}$. By induction reduce to $k = 1$:

$$\begin{array}{lll} u \in W^{k,p} & \Rightarrow & u, \partial_j u \in W^{k-1,p} \\ & \Rightarrow & \text{(induction) } u, \partial_j u \in L^{p'} \qquad 0 > k - 1 - \frac{n}{p} = 0 - \frac{n}{p'} \\ & \Rightarrow & u \in W^{1,p'} \\ & \Rightarrow & (k=1) \ u \in L^q \qquad 0 > 1 - \frac{n}{p'} = k - \frac{n}{p} \geq 0 - \frac{n}{q} \end{array}$$

To prove $W^{1,p} \hookrightarrow L^q$ seek

$$\boxed{\|u\|_q \leq c\|du\|_p \qquad \forall u \in C^\infty_c(X) \qquad \text{(fails for } u = 1) \qquad (*)}$$

Sketch: You start from fund. thm of calculus "$u(x) = \int_{-\infty}^{x_i} \partial_i u(\cdots)\, dx_i$" ($p = 1$), then everything else[11] is repeated integrations and Hölder's inequalities. For general p, you just use clever exponents.

Fact. can extend $u \in W^{1,p}(X)$ to a compactly supported $\overline{u} \in W^{1,p}(\mathbb{R}^n)$ in a way that $\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq c\|u\|_{W^{1,p}(X)}$. Then approximate by $C^\infty_c(\mathbb{R}^n)$ and use $(*)$. $\square$

Rmks.

(1) *Can replace W by W_0 (then no conditions on ∂X are needed).*
(2) *$(*)$ holds $\forall u \in W^{1,p}_0(X)$ $(0 > 1 - \frac{n}{p} > 0 - \frac{n}{q})$, a version of* Poincaré's ineq.
(3) *$kp > n$ is wonderful since $W^{k,p} \subset C^0$, so you can represent elements as continuous functions, avoiding the Cauchy rubbish.*

4.4. **Derivatives on $W^{k,p}$.**
Method 1: via completions

$$\partial_s : (C^\infty, \|\cdot\|_{k,p}) \to (C^\infty, \|\cdot\|_{k-1,p})$$
$$\partial_s : W^{k,p} \to W^{k-1,p}, [u_n] \to [\partial_s u_n]$$

Method 2: for $p = 2$, use the Fourier transform to replace ∂^I by multiplication by x^I (up to a constant factor). See Hwk 11.

Method 3: Using weak derivatives
First we want to avoid completions, and work with actual functions:

$$L^p(X) = \{\text{Lebesgue measurable } u : X \to \mathbb{R} \text{ with } \|u\|_p < \infty\} \Big/ \underset{\text{almost everywhere}}{u \sim v \text{ if } u = v}$$

[10] using $\|u\|_{C^j} = \sum_{|I|\leq j} \sup |\partial^I u|$ on $C^j(\overline{X})$: call C^j_b the subset of u's with bdd $\|u\|_{C^j}$.
[11] If you're curious, see Evans, *Partial Differential Equations*, p.263.

For our purposes, we don't need a deep understanding of measure theory, just a vague nod: *Lebesgue measure* is a good notion of volume for certain subsets of $\mathbb{R}^n$. These subsets are called *measurable.* For example open subsets and closed subsets. The notion of volume for cubes and balls is what you think, and there are various axioms, the most important of which is: the volume of a countable disjoint union of subsets is the sum of the individual volumes. Define:

$$f \text{ is } \textit{measurable} \text{ if } f^{-1}(\text{any open set}) \text{ is measurable.}$$

Examples. Continuous functions, since $f^{-1}(\text{open})$ is open. Step functions (for example $f = 1$ on some open set, $f = 0$ outside it). Also: can add, scale, multiply, take limits of measurable fns to get measurable fns.[12]

Convention. If $u \sim$ continuous fn, then we always represent u by the cts fn!

Fact. *The above $(L^p(X), \|\cdot\|_p)$ is complete and $C^\infty(X) \subset L^p(X)$ dense, so $L^p(X) \cong$ completion of $(C^\infty(X), \|\cdot\|_p)$ (as usual, only allow smooth u with $\|u\|_p < \infty$).*

Def. *$f_I \in L^p(X)$ is an I-th weak derivative of f if $\forall \phi \in C_c^\infty(X)$,*

$$\int_X f_I \cdot \phi \, dx = (-1)^{|I|} \int_X f \cdot \partial^I \phi \, dx.$$

For smooth f this is just integration by parts with $f_I = \partial^I f$.

Exercise. weak derivatives are unique if they exist. So just write $f_I = \partial^I f$.

Key Fact. $\boxed{\text{if } f_I \text{ is cts, then the usual } \partial^I f \text{ exists and equals } f_I.}$

See Lieb & Loss, *Analysis*, 2nd ed. Non-examinable: If $u \in W^{k,p}_{loc}$ then $u \in W^{1,1}_{loc}$ by Lemma 4.2 (*loc* gives finite vol), hence the FTC holds (L&L p.143): $u(x+y) - u(x) = \int_0^1 y \cdot \nabla u(x+ty)\, dt$ for a.e. x, all small y. Their proof shows that this is true for all x if $u, \nabla u$ are continuous. The key fact is proved in L&L p.145. Example: Suppose $u \in W^{1,2}(\mathbb{R}, \mathbb{R}^m)$ with cts weak $\partial_s u$, then $\frac{u(s+y)-u(s)}{y} = \int_0^1 \partial_s u(s+ty)\, dt \to \partial_s u(s)$ as $y \to 0$ by cty, so u is C^1 with deriv = weak deriv. Proof of FTC for $u \in W^{1,2}(\mathbb{R}, \mathbb{R})$: for $\phi \in C_c^\infty(\mathbb{R}, \mathbb{R})$, write $\widetilde{\phi}(s) = \phi(s - ty)$.

$$\begin{aligned} \textstyle\int \phi(s)(\int_0^1 y \cdot \partial_s u(s+ty)\, dt)\, ds &= \textstyle\int_0^1 y(\int \phi(s) \cdot \partial_s u(s+ty)\, ds) dt = -\int_0^1 \int y \cdot \partial_s \widetilde{\phi}(s) \cdot u(s)\, ds\, dt \\ &= \textstyle\int \int_0^1 \partial_t \widetilde{\phi}(s)\, dt\, u(s)\, ds = \int \phi(s-y) u(s)\, ds - \int \phi(s) u(s)\, ds = \int \phi(s)(u(s+y) - u(s))\, ds. \end{aligned}$$

Hence $\int_0^1 y \cdot \partial_s u(s+ty)\, dt = u(s+y) - u(s)$ for a.e. s (all s if $u, \partial_s u$ cts (u is cts by Sobolev)).

Rmks.

(1) *Weak derivatives behave as you expect:*

$$\partial^I : W^{k,p} \to W^{k-|I|,p} \quad \textit{is linear.}$$

Also $\phi \in C_c^\infty$, $u \in W^{k,p} \Rightarrow \phi \cdot u \in W^{k,p}$ with $\partial^I(\phi \cdot u) =$ Leibniz formula.

(2) *Observe:*

$$\begin{aligned} u \in W^{k,p}(X) \quad &\Rightarrow \quad u = (\|\cdot\|_{k,p}\text{-}\textit{Cauchy sequence of smooth } u_n : X \to \mathbb{R}) \\ &\Rightarrow \quad \partial^I u_n \textit{ are } \|\cdot\|_p\text{-}\textit{Cauchy } \forall |I| \le k \\ &\Rightarrow \quad \partial^I u_n \to u_I \textit{ in } L^p, \textit{ some } u_I \in L^p(X)\, (\textit{by completeness of } L^p). \end{aligned}$$

But $\int \partial^I u_n \cdot \phi = (-1)^I \int u_n \cdot \partial^I \phi$, take $n \to \infty$, deduce $u_I =$ weak derivs!

Thm. *Define*

$$W^{k,p}(X) = \{u \in L^p(X) : u \textit{ has weak derivatives } \partial^I u \in L^p(X), \forall |I| \le k\}$$

then $C^\infty(\overline{X}) \cap W^{k,p}(X) \subset W^{k,p}(X)$ dense, so $W^{k,p}(X) \cong$ completion construction.

Pf uses a standard method to smoothly approximate measurable fns: **mollifiers**.[13]

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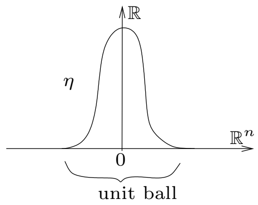
[12] *Non-examinable:* Any measurable fn is a limit of simple fns. Simple fns are linear combinations of characteristic fns χ_S of measurable subsets S ($\chi_S(s) = 1\, \forall s \in S$, else $\chi_S = 0$).

[13] An explicit η is the following: $c \cdot \exp(\frac{1}{|x|^2 - 1})$ for $|x| \le 1$, and 0 otherwise.



$\eta : \mathbb{R}^n \to \mathbb{R}$ smooth bump function, normalized so that $\int_{\mathbb{R}^n} \eta \, dx = 1$.

For $\varepsilon > 0$, define[14] $\boxed{\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \cdot \eta(\frac{x}{\varepsilon})}$

Observe: $\eta_\varepsilon \in C^\infty(\mathbb{R}^n, \mathbb{R})$, $\operatorname{supp} \eta_\varepsilon \subset \varepsilon$-ball, and $\int_{\mathbb{R}^n} \eta_\varepsilon \, dx = 1$.

For $u : X \to \mathbb{R}$ in $L^1_{loc}(X)$, define ε-mollification as a *convolution*:

$$\begin{aligned} u_\varepsilon(x) &= (\eta_\varepsilon * u)(x) \\ &= \int_X \eta_\varepsilon(x-y)\, u(y)\, dy \\ &= \int_{\varepsilon\text{-ball}} \eta_\varepsilon(y)\, u(x-y)\, dy \end{aligned}$$

defined for $x \in X_\varepsilon = \{$points of X at distance $> \varepsilon$ from $\partial X\}$.

Fact.[15]

(1) *$u_\varepsilon(x)$ only depends on values of u near x* (*Pf. 2nd integral.*)
(2) *$u_\varepsilon \in C^\infty(X_\varepsilon)$* (*Pf. differentiate 1st integral.*)
(3) *$u_\varepsilon(x) \to u(x)$ for almost any x as $\varepsilon \to 0$*
(4) *u continuous $\Rightarrow u_\varepsilon \to u$ uniformly on compacts (hence $C^\infty \subset C^0$ is dense)*
(5) *$u \in L^p_{loc}(X) \Rightarrow u_\varepsilon \to u$ in $L^p_{loc}(X)$*

Cor. *$u \in W^{k,p}(X) \Rightarrow u_\varepsilon \to u$ in $W^{k,p}_{loc}(X)$*

Proof. Easy computation:

$$\partial^I u_\varepsilon = \eta_\varepsilon * \partial^I u \qquad (\text{in } X_\varepsilon)$$

but $\partial^I u \in L^p(X)$, so by (5), $\eta_\varepsilon * \partial^I u \to \partial^I u$ in L^p_{loc}. □

This corollary essentially implies the theorem by a clever[16] partition of unity argument (non-examinable).

4.5. Elementary proof of Sobolev/Rellich for $W^{1,2}$.

Theorem 1. *$W^{1,2}(\mathbb{R}) \overset{bdd}{\hookrightarrow} C^0_b(\mathbb{R}) = \{bdd\ cts\ \mathbb{R} \to \mathbb{R}\}$, and $W^{1,2}(\mathbb{R}) \overset{cpt}{\to} C^0([-S, S])$.*

Proof. For $u \in W^{1,2}$, pick $u_n \in C^0 \cap W^{1,2}$ converging to u in $W^{1,2}$ (by mollification, $C^0 \cap W^{1,2} \subset W^{1,2}$ is dense). So u_n is $W^{1,2}$-bdd and by Cauchy-Schwarz

$$|u_n(b) - u_n(a)| \leq \int_a^b |\partial_s u_n|\, ds \leq \sqrt{|b-a|} \cdot \|u_n\|_{1,2} \leq \text{const} \cdot \sqrt{|b-a|} \qquad (*)$$

so u_n is equicts. To check u_n is equibdd, suppose $u_n(a)$ is unbdd (fixed a). By $(*)$

$$\left| \min_{b \in [a-1, a+1]} u_n(b) - u_n(a) \right| \leq C$$

so that minimum is also unbdd. So u_n is L^2-unbdd, contradicting $W^{1,2}$-bdd.

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[14] as $\varepsilon \to 0$, intuitively "$\eta_\varepsilon \to$Dirac delta".

[15] If you're curious: Evans, *Partial Differential Equations*, p.630.

[16] If you're curious: Evans, *Partial Differential Equations*, p.251-254. The Corollary gives the Thm for $C^\infty(X)$, and to get $C^\infty(\overline{X})$ one needs a little care near the boundary ∂X because the convolution requires having an ε-ball around x inside the domain. The fix is to locally (on a small open $V \subset X$) translate u: $\widetilde{u}(x) = u(x - c\varepsilon\vec{n})$ where $\vec{n}$ is the outward normal along ∂X and c is a large constant. Then $\eta_\varepsilon * \widetilde{u} \in C^\infty(\overline{V})$ cges to u in $W^{k,p}(V)$.

By Arzela-Ascoli, there is a subsequence $u_n|_{[-S,S]} \to v$ in $C^0[-S,S]$, so also in $L^2[-S,S]$, so $v = u|_{[-S,S]}$, so u is cts since S was arbitrary.

Need to check u is C^0-bounded. As in $(*)$, $|u(s+1)-u(s)| \leq \|u|_{[s,s+1]}\|_{1,2}$, so

$$|u(s+m)-u(s)| \leq \|u|_{[s,s+1]}\|_{1,2}+\cdots+\|u|_{[s+m-1,s+m]}\|_{1,2} = \|u|_{[s,s+m]}\|_{1,2} \leq \|u\|_{1,2}$$

so u is bdd at $\pm\infty$, hence bdd on $\mathbb{R}$ by cty. $\square$

4.6. $W^{k,p}$ **for manifolds.** Let N^n be a compact mfd and M^m any mfd.

$W^{k,p}(N) = W^{k,p}(N,\mathbb{R})$ and $W^{k,p}(N,M)$ are the completion of $C^\infty(N)$ and $C^\infty(N,M)$ w.r.t. the $\|\cdot\|_{k,p}$ norm defined below. Equivalently, they are the space of measurable functions/maps[17] which are k-times weakly differentiable (in the charts below) and which have bounded $\|\cdot\|_{k,p}$-norm.

Def. *$W^{k,p}(N,\mathbb{R}^m)$ for N^n compact mfd: pick a finite cover by charts*[18]

$$\varphi_i : (\textit{ball } B_i \subset \mathbb{R}^n) \to U_i \subset N$$

For $u : N \to \mathbb{R}^m$, define $\boxed{\|u\|_{k,p} = \sum \|u \circ \varphi_i\|_{W^{k,p}(B_i,\mathbb{R}^m)}}$

$W^{k,p}(N,M)$, any mfd M^m: fix smooth embedding $j : M \hookrightarrow \mathbb{R}^a$. For $u : N \to M$ let[19]

$$\|u\|_{k,p} = \|j \circ u\|_{W^{k,p}(N,\mathbb{R}^a)}.$$

Rmk.

(1) *N compact $\Rightarrow$ get equivalent norms if change charts*

(2) *$X, Y \subset \mathbb{R}^n$ open, $k \geq 1$, call $\phi : X \to Y$ a C^k-diffeo if: ϕ is a homeomorphism, $\phi \in C^k(\overline{X},\overline{Y})$, $\phi^{-1} \in C^k(\overline{Y},\overline{X})$ and both have bdd C^k-norm.*

Fact. *$W^{k,p}(Y) \xrightarrow{\circ\phi} W^{k,p}(X)$ is bdd with bdd inverse.*

Cor. *N compact $\Rightarrow$ get equivalent norm if change φ_i, U_i.*

(3) *$\phi : X \to Y$ has bdd C^k-norm $\Rightarrow W^{k,p}(N,X) \xrightarrow{\phi\circ} W^{k,p}(N,Y)$ bdd*

Rmk. *just bound $\phi \circ u$ in terms of $\|\phi\|_{k,p}$, $\|u\|_{C^k}$. If you wanted to bound $\phi \circ u$ in terms of $\|\phi\|_{k,p}$, $\|u\|_{k,p}$, then even for smooth ϕ you need $kp > n$.*

Cor. *M compact $\Rightarrow$ choice of j does not matter (for non-cpt M it matters)*

4.7. $W^{k,p}$ **for vector bundles.** For a vector bundle $E \to N$,

$$W^{k,p}(E) = \{W^{k,p} \text{ sections } u : N \to E\}$$

In this case, you can avoid picking j:

$$B_i \times \mathbb{R}^r \overset{\varphi_i}{\cong} U_i \times \mathbb{R}^r \overset{\text{triv}}{\cong} E|_{U_i}$$

view $u \circ \varphi_i$ as a map $B_i \to \mathbb{R}^r$

$$\|u\|_{k,p} = \sum \|(\rho_i \cdot u) \circ \varphi_i\|_{W^{k,p}(B_i,\mathbb{R}^r)}$$

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[17] $W^{k,p}(N,M) \subset W^{k,p}(N,\mathbb{R}^a)$, the $u : N \to \mathbb{R}^a$ with $u(n) \in M \subset \mathbb{R}^a$ for almost every $n \in N$.

[18] strictly speaking these are *parametrizations*: they go from $\mathbb{R}^n$ to N. If you want charts $\varphi_i : U_i \to \mathbb{R}^n$, then you need bump functions ρ_i subordinate to the U_i: $\sum \|(\rho_i \cdot u) \circ \varphi_i^{-1}\|_{W^{1,2}(\mathbb{R}^n,\mathbb{R})}$.

[19] Using charts on M would be a bad idea: think about why that would not work.

Alternatively, pick: a Riem metric g_N on N, a metric g_E on E (smoothly varying inner product for each fibre), and a connection ∇ on E. Then define:[20]

$$\boxed{\|u\|_{k,p} = \sum_{i\le k}\left(\int_N |\nabla^i u|^p \operatorname{vol}_N\right)^{1/p}}$$

Lemma 2. *N compact $\Rightarrow$ those two definitions give equivalent norms.*

Proof. Choice of local trivializations doesn't matter since they change by multiplication by a smooth matrix-valued map (use Rmk 3 above).

Pick local trivializns using smooth local *orthonormal* sections. So $|\cdot|$ differs from $|\cdot|_{\mathbb{R}^a}$ only by use of g_N^* in $\Omega^i(N)$ directions. So get bounds since N is compact.

Locally $\nabla = d+A$ (A local section of $\operatorname{End}(E)$), hence can bound $u,\ldots,\nabla^{i-1}u, \nabla^i u$ in terms of $\|A\|_\infty$, u, $\partial^I u$ ($|I|\le i$). Vice-versa can bound $\partial^I u$ in terms of $\|A\|_\infty$, $\nabla^i u$ ($i\le |I|$) by the triangle inequality. □

4.8. **Sobolev theorems for manifolds.** For a compact mfd N, any mfd M:

$$\boxed{\begin{array}{lll} L^p(N) \overset{\text{bdd}}{\hookrightarrow} L^q(N) & \text{for } p\ge q \quad (\text{since } \operatorname{vol}(N)<\infty) & \\ W^{k,p}(N,M) \overset{\text{bdd}}{\hookrightarrow} W^{k',p'}(N,M) & \text{for } \begin{cases} k\ge k' \\ k-\frac{n}{p}\ge k'-\frac{n}{p'} \\ {\scriptstyle (p'\ne\infty \text{ if } kp=n)} \end{cases} & (\text{compact if strict } >\text{'s}) \\ W^{k,p}(N,M) \overset{\text{bdd \& cpt}}{\hookrightarrow} C^{k'}(N,M) & \text{for } k-\frac{n}{p}>k' & \end{array}}$$

Warning. Fails for non-compact N, unless you have control of the geometry at ∞: for example for $N=\mathbb{R},\mathbb{R}^n$ the above still holds.

Def. $W^{k,p}_{loc}(N,M) = \{u : N\to M :\ u|_C \in W^{k,p}(C,M),\ \forall C \Subset N\}$

Warning. the $W^{k,p}_{loc}$ are not normed, but they are complete metric spaces with the topology: $u_n\to u$ in $W^{k,p}_{loc} \Leftrightarrow u_n|_C \to u|_C$ in $W^{k,p}(C,M)\,\forall C\Subset N$.
Exercise. $u\in W^{k,p}_{loc} \Leftrightarrow \exists u_n\in C^\infty_c$, $u_n\to u$ in $W^{k,p}_{loc}$. So $W^{k,p}_{loc}\cong$ completion defn.

Cor. *Sobolev embeddings hold for*[21] $W_{loc}, L_{loc}, C_{loc}$ *even for non-compact N.*

Proof. $u\in W^{k,p}_{loc}(N,M) \Rightarrow u|_C\in W^{k,p}(C,M) \Rightarrow u|_C \in W^{k',p'}(C,M) \Rightarrow u\in W^{k',p'}(N,M)$ □

[20] where $\nabla^0 u = u$, $\nabla^i : C^\infty(E)\to\Omega^i(N)\otimes C^\infty(E)$ (extending ∇ to higher forms by Leibniz: $\nabla(\omega\otimes s) = d\omega\otimes s+\omega\otimes\nabla s$), and $\operatorname{vol}_N = \sqrt{|\det g_N|}\,dx_1\wedge\cdots\wedge dx_n$, and the norm in the integral combines the norm from g_E on E and the norm from the dual metric g_N^* on T^*N (which induces a metric on the exterior product $\Lambda^i T^*N$ - explicitly, use g_N to locally define an orthonormal frame for TN by Gram-Schmidt, declare the dual of that to be an o.n. frame for T^*N, this determines g_N^*, and taking ordered i-th wedge products you declare what an o.n. frame for $\Lambda^i T^*N$ is).

[21] C^k_{loc} just means C^k-convergence on compact subsets.