LECTURE 12.

PART III, MORSE HOMOLOGY, 2011

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4.9. **Sobolev setup for the transversality theorem.** Let M be a closed manifold, $f : M \to \mathbb{R}$ a Morse function, and fix critical points $p \neq q \in \mathrm{Crit}(f)$ and a reference metric g_M for M (all norms will refer to g_M, not the variable metric g).

Consider the bundle mentioned in 4.0:

$$\begin{tikzcd} E \arrow[d] \\ U \times M \arrow[u, bend right, "F(u,g) = \partial_s u + f'_g(u)"'] \end{tikzcd}$$

where:[1]

$$\boxed{\begin{aligned} G = &\quad \{C^k\text{-metrics on } M\} \ \ (\text{fix } k \geq 1) \\ U = &\quad \{u \in W^{1,2}_{loc}(\mathbb{R}, M) : u(s) \to p, q \text{ as } s \to -\infty, +\infty \text{ and for large } \mathcal{S} \text{ we have } : \\ &\qquad u(s) = \exp_p(\xi(s)) \text{ for } s \leq -\mathcal{S}, \text{some } \xi \in W^{1,2}((-\infty, -\mathcal{S}), T_pM) \\ &\qquad u(s) = \exp_q(\xi(s)) \text{ for } s \geq +\mathcal{S}, \text{some } \xi \in W^{1,2}((+\mathcal{S}, \infty), T_qM)\} \\ E = &\quad \{L^2\text{-vector fields along the paths } u \in U\} \end{aligned}}$$

By Sobolev, $W^{1,2}_{loc} \subset C^0_{loc}(\mathbb{R}, M)$, so the $u \in U$ are continuous, and requiring convergence to p, q at $\pm\infty$ makes sense.

E is a vector bundle over $U \times G$ with fibre $L^2(u^*TM)$, the L^2-sections of the pull-back bundle $u^*TM \to \mathbb{R}$ whose fibre is $(u^*TM)_s = T_{u(s)}M$ over $s \in \mathbb{R}$.

Lemma. *G is a smooth Banach manifold.*

Proof. $G \subset C^k(Sym^2(T^*M))$ is an open subset of the space[2] of symmetric 2-forms on TM, since positive definiteness is an open condition. We have a regular retraction $\pi : U \to M$ of a tubular nbhd of $j : M \hookrightarrow \mathbb{R}^a$ (so $\pi \circ j = id$). Recall by 1.6, that for a Banach space B and a closed subset $S \subset B$,

$$S \text{ smooth retract of an open nbhd of } S \subset B \Rightarrow S \subset B \text{ is a submfd.}$$

Also recall that for any map $\varphi : A \to B$ of mfds, g metric on B, the pull-back metric is defined by $(\varphi^* g)_a(v, w) = g_{\varphi a}(d\varphi \cdot v, d\varphi \cdot w)$ for $v, w \in T_aA$.

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[1]**Rmk.** *We cannot just say $U \subset W^{1,2}(\mathbb{R}, M)$: Sobolev spaces don't make sense for non-compact domains (unless you are considering sections of a bundle). Example: the constant $u : \mathbb{R} \to \{p\} \in M$ has $\int_{\mathbb{R}} |u(s)|^2\, ds = |p|^2 \cdot \int_{\mathbb{R}} 1\, ds = \infty$ for $|p| \neq 0$ the norm of $p \in M \subset \mathbb{R}^a$. Similarly, continuous u converging to p, q have infinite L^2-norm. So our Sobolev spaces would be empty! We want each $p \in Crit(f)$ to be considered to have zero norm, for that reason we chose (canonical, using* exp*) charts around the critical points and require u to be $W^{1,2}$ in that chart.*

[2]Example: $g = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right)$ is the symmetric form $dx^{\otimes 2} + 2dx \otimes dy + 2dy \otimes dx + 3dy^{\otimes 2}$, so $g(\partial_x, \partial_y) = 2$.

First note that

$$S = C^k(Sym^2(T^*M)) \hookrightarrow C^k(Sym^2(T^*U)), g \mapsto \pi^* g$$

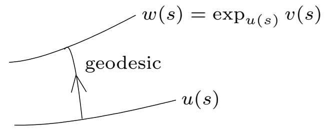
is a closed subset (injective since $j^*\pi^*g = (\pi j)^*g = g$). Secondly

$$C^k(Sym^2(T^*U)) \to S, g \mapsto \pi^* j^* g$$

is a retraction (since $\pi^*j^*\pi^*j^*g = \pi^*(\pi j)^*j^*g = \pi^*j^*g$). Finally, both these maps are smooth since linear in g. Thus, by the above result, S is a smooth manifold, hence also the open subset G is a smooth Banach manifold. $\square$

Lemma. *U is a smooth Banach manifold modeled on $W_0^{1,2}(\mathbb{R}, \mathbb{R}^m)$.*

Proof. The reference metric defines and exp map, and by Cor 0.7: any C^0-close continuous paths u, w are homotopic through geodesic arcs joining $u(s), w(s)$.

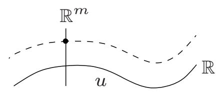


Let $\varepsilon > 0$ be as in that Corollary. For each *smooth* $u \in U$, define

$$\boxed{W = \{\exp_{u(s)} v(s) : v \in W_0^{1,2}(u^* D_\varepsilon TM)\}}$$

(so in particular, $v(s) \to 0$ as $s \to \pm\infty$). You can easily check that $W \subset U$, by construction (this crucially uses the fact that $u, \exp$ are smooth maps, so $\exp_u v$ is as smooth as v is). By Cor 0.7, and the density of C^∞ maps inside $W^{1,2}$ maps, any $w \in U$ is within ε-distance of some smooth $u \in U$, therefore these W's cover U.

Let ∇ be the Levi-Civita connection for g_M, then by parallel translation



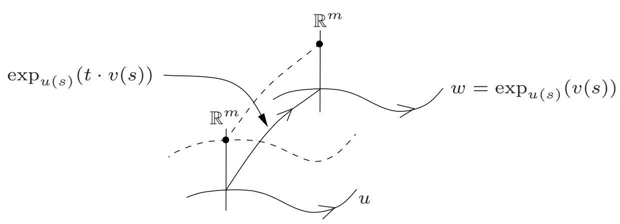
$$\boxed{u^*TM \cong \mathbb{R} \times \mathbb{R}^m}$$

so $W_0^{1,2}(u^*TM) \cong W_0^{1,2}(\mathbb{R}, \mathbb{R}^m)$, which is a Banach space. Thus the v are local coordinates for U near u. You can easily check that transition maps on overlaps are smooth, since they involve *smooth* maps $u_0, u_1, \exp$. $\square$

Rmk. *For a Banach manifold M modeled on B, the tangent space is $T_mM = B$ (compare with $T_x\mathbb{R}^n \equiv \mathbb{R}^n$). For a chart $\varphi : M \supset U \to B$ define $TM|_U = \varphi(U) \times B$. For another chart $\psi : U' \to B$, the transition $\varphi(U \cap U') \times B \to \psi(U' \cap U) \times B$ is $(x, b) \mapsto (x, d_x(\psi \circ \varphi^{-1}) \cdot b)$. In our example:* $\boxed{T_v W_0^{1,2}(u^* D_\varepsilon TM) \equiv W_0^{1,2}(u^*TM)}$

Lemma. *E is a smooth Banach vector bundle with fibre $L^2(\mathbb{R}, \mathbb{R}^m)$.*

Proof. Consider parallel transport along geodesics:



$$P_v : u^*TM \xrightarrow{\sim} (\exp_u v)^*TM = w^*TM$$

Recall this map is smooth (since linear) and is an isometry. Now, consider its dependence on v:

$$P : W_0^{1,2}(u^*D_\varepsilon TM) \times u^*TM \xrightarrow{\sim} \bigcup_{w\in W} w^*TM.$$

This is again smooth.[3]

Finally, consider parallel transporting L^2-vector fields over u:

$$P : W_0^{1,2}(u^*D_\varepsilon TM) \times L^2(u^*TM) \xrightarrow{\sim} L^2(\bigcup_{w\in W} w^*TM) = E|_W.$$

This is well-defined since parallel transport is an isometry (so L^2 sections map to L^2 sections), and is invertible by doing parallel transport backwards. It is smooth for the same reasons as before. As above, we can trivialize: $L^2(u^*TM) \cong L^2(\mathbb{R}, \mathbb{R}^m)$. Thus we have obtained a trivialization of $E|_W$:

$$E|_W \cong W \times L^2(\mathbb{R}, \mathbb{R}^m)$$

and two trivializations differ by smooth maps since $u_0, u_1, \exp$ are smooth. $\square$

[3] *Non-examinable:* The ODE you solve is $\partial_t \vec{x}(t) = -A_{\exp_{u(s)}(tv(s))}((d\exp_{u(s)})_{tv(s)} \cdot v(s)) \cdot \vec{x}(t)$, $\vec{x}(0) \in T_{u(s)}M$ (where $s \in \mathbb{R}$ is fixed, $t \in [0,1]$ varies). Change $v(s)$ to $v(s) + \lambda\vec{v}(s)$: observe that $\vec{x}(1)$ is smooth in $\lambda \in \mathbb{R}$ because solutions of ODEs depend smoothly on initial conditions.