LECTURE 13.

PART III, MORSE HOMOLOGY, 2011

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4.9. **Sobolev setup for transversality theorem (continued).**

$$\begin{tikzcd} E \arrow[d] \\ U \times M \arrow[u, bend right, "{F(u,g) = \partial_s u + f_g'(u)}"'] \end{tikzcd}$$

where f_g' is ∇f calculated for g: $g(f_g', \cdot) = df$.

Lemma. *F is a well-defined section.*

Proof. The weak derivative $\partial_s : W^{1,2} \to L^2$ is well-defined so only f_g' may be an issue. Since f_g' is C^k, it is L^2 on compacts, so we just need to check that $f_g'(u)$ is L^2 near the ends. Locally near a critical point $p = (x = 0)$:

$$|f_g'(x)| \leq c \cdot |x| \text{ by Taylor, since } f_g'(p) = 0.$$

Hence $|f_g'(u)| \leq c \cdot |u|$, so $f_g'(u)$ is L^2 near the ends since u is L^2. □

Lemma. *F is C^k.*

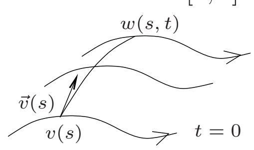
Proof. Differentiating in the g direction: only f_g' contributes, and

$$df = g(f_g', \cdot)$$

so locally $f_g' = g^{-1} \cdot \partial f$. This is linear in g^{-1}, hence smooth in g^{-1}. Finally, inversion is smooth on non-singular matrices. So f_g' is smooth in g.

Differentiating in the direction $\vec{v} \in W_0^{1,2}(u^*TM) \equiv T_v W_0^{1,2}(u^* D_\varepsilon TM)$ (where u is smooth):

$$\begin{aligned} D_v F \cdot \vec{v} &= \nabla_t|_{t=0}(\partial_s w - f_g'(w)) && \text{(see Hwk 12)} \\ &\quad \text{where } w : \mathbb{R} \times [0,1] \to M && w(s,t) = \exp_{v(s)}(t\vec{v}(s)) \end{aligned}$$



$$\begin{aligned} &= (\nabla_s \partial_t w - \nabla_t f_g'(w))_{t=0} && \text{(using } \nabla \text{ torsion-free (see Hwk 12))} \\ &= \nabla_s \vec{v} - (\nabla_{\vec{v}} f_g')_v \end{aligned}$$

Note: $\nabla_s = \partial_s + A(\partial_s v)\cdot$ is linear in v so C^∞ in v, $(\nabla_{\vec{v}} f_g')_v = (d(f_g') \bullet + A(\bullet) \cdot f_g') V$: the first term is C^{k-1}, $A(\bullet)$ is C^∞, f_g' is C^k, so $(\nabla_{\vec{v}} f_g')_v$ is[1] C^{k-1} in v. □

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[1]The key point is that we never differentiate v in s.

4.10. **Transversality Theorem.**

Lemma. $F(u,g)=0 \Rightarrow u \in C^{k+1}$ *(motto: u is "as smooth as" g)*

Proof. The weak derivative $\partial_s u = f_g'(u)$ is continuous since u is cts.[2] Recall that if the weak derivative is cts then it equals the usual derivative. Now bootstrap.[3] $\square$

Conditions of Thm 2.5
$\forall\, F(u,g)=0$:
i) $D_{(u,g)}F : T_{(u,g)}(U \times G) \to E_{(u,g)}$ surjective;
ii) $D_u F_g : T_u U \to E_{(u,g)}$ Fredholm of index $< k$.

Claim 1. (ii) holds and index $= |p|-|q|$ (so pick $k > |p|-|q|$)
Claim 2. (i) holds

We will prove the Claims later. First we mention the consequence:

Transversality Theorem
For generic C^k-metrics g,

$$W^k(p,q) = \{C^k\text{-flowlines } u \text{ of } -\nabla f \text{ from } p \text{ to } q\} = F_g^{-1}(0)$$

is a C^k-submanifold of U with

$$\dim W^k(p,q) = \operatorname{index} F_g = |p|-|q|$$

and tangent space

$$T_u W^k(p,q) = \ker(D_u F_g : T_u U \overset{\text{surj}}{\to} E_u)$$

Proof. Theorem 2.5 and Claims 1 & 2 above. $\square$

Cor. *Can take $k=\infty$*

Proof. Pick C^k metric g satisfying $\pitchfork$,
$\Rightarrow \exists C^\infty$ g' close to g
$\Rightarrow g'$ satisfies $\pitchfork$ since transversality is an open condition (regularity is open).
Hwk 13: C^∞-metrics g satisfying $\pitchfork$ are generic. $\square$

$$\Rightarrow \boxed{\begin{array}{l} \text{for a generic smooth metric,} \\ \quad W(p,q) = W^\infty(p,q) \text{ is a smooth mfd of dim} = |p|-|q| \\ \quad \mathcal{M}(p,q) = W(p,q)/\mathbb{R} \text{ is a smooth mfd of dim} = |p|-|q|-1 \end{array}}$$

Rmk. *Hwk 13 proves that the quotient $\mathcal{M}(p,q)$ really is a smooth mfd.*

Rmk. *Hwk 19 proves that the transversality condition for $W(p,q)$, that is the surjectivity of the linearization $D_u F_g$ at zeros of F_g, is equivalent to the condition that g is Morse-Smale for f. So* $\boxed{\textit{"generic } g\textit{" is equivalent to saying "}g\textit{ is Morse-Smale"}}$

[2] *Non-examinable:* this is an example of the elliptic regularity theorem: ∂_s is an elliptic operator of order 1, and so weak solutions of $\partial_s u = -f_g'(u)$ are as regular as f_g' plus order, so u is C^{k+1}.

[3] We proved u is C^1, so $\partial_s u = f_g'(u)$ is C^1 (since f_g' is C^k and u is C^1) so u is C^2 etc.

4.11. **Hilbert spaces tricks.** $L : A \to B$ bounded linear, A Banach, B Hilbert.

Lemma 1 *If* $\operatorname{im}(L)$ *is closed, then*

$$\operatorname{coker} L \cong (\operatorname{im} L)^{\perp} = \{b \in B : \langle La, b\rangle = 0 \ \forall a \in A\}.$$

Proof. In general, if $V \subset B$ closed subspace, then $B = V \oplus V^{\perp}$, so $V^{\perp} \cong B/V$. □

Warning. $C^{\infty} \subset L^2$ dense (non-closed) subspace: $(C^{\infty})^{\perp} = 0$, but $C^{\infty} \neq L^2$.

Def. *Define the* adjoint $L^* : B \to A$ *of* $L : A \to B$*, where* A, B *Hilbert, by*

$$\langle La, b\rangle_B = \langle a, L^* b\rangle_A \qquad \forall a, b$$

(easy exercise: $\exists$ *unique bounded linear such* L^**).*

Lemma 2 $(\operatorname{im} L)^{\perp} \cong \ker L^*$.

Proof. $b \perp \operatorname{im} L \Leftrightarrow \langle La, b\rangle = 0 = \langle a, L^* b\rangle \ \forall a \Leftrightarrow L^* b = 0$. □

$L : A = A_1 \times A_2 \to B$, A_1 Hilbert, A_2 Banach, B Hilbert.
$L(a_1, a_2) = D(a_1) + P(a_2)$
$D : A_1 \to B$, $P : A_2 \to B$ bounded linear (think of P as "perturbation").

Lemma 3 $\operatorname{im} L$ *closed* $\Rightarrow \operatorname{coker} L \subset \ker(D^* : B \to A_1) \cap (\operatorname{im} P)^{\perp}$.

Proof. $\operatorname{im} D \subset \operatorname{im} L$
$\Rightarrow \ker D^* \overset{\text{by } 2}{\cong} (\operatorname{im} D)^{\perp} \supset (\operatorname{im} L)^{\perp} \overset{\text{by } 1}{\cong} \operatorname{coker} L$
$\Rightarrow \operatorname{coker} L \perp \operatorname{im} L \supset \operatorname{im} P$. □

Lemma 4 D Fredholm $\Rightarrow \operatorname{im} L$ closed.

Proof. $B = \operatorname{im} D \oplus C$, $C =$ complement (finite dimensional).
$\operatorname{im} L = \operatorname{im} D \oplus (C \cap \operatorname{im} L)$ (equality holds since $\operatorname{im} D \subset \operatorname{im} L$)
Finally: $\operatorname{im} D$ closed, and $C \cap \operatorname{im} L$ is finite dimensional hence closed. □

4.12. **Claim 1** $\Rightarrow$ **Claim 2.** We will apply Lemma 3 to:

$$\underbrace{(D_{(u,g)} F)}_{L} \cdot (\vec{u}, \vec{g}) \quad = \quad \underbrace{D_u F_g}_{D} \cdot \vec{u} - \underbrace{D_{(u,g)} f'_g}_{P} \cdot \vec{g}$$

$A_1 = W_0^{1,2}(u^* TM) \cong W_0^{1,2}(\mathbb{R}, \mathbb{R}^m) \quad \langle a, a'\rangle_{1,2} = \int_{\mathbb{R}} g_M(a, a')\, ds + \int_{\mathbb{R}} g_M(\partial_s a, \partial_s a')\, ds$
$A_2 = T_g G$
$B = E_u = L^2(u^* TM) \cong L^2(\mathbb{R}, \mathbb{R}^m) \quad \langle b, b'\rangle_{L^2} = \int_{\mathbb{R}} g_M(b, b')\, ds.$

Where D is Fredholm by Claim 1.

Rmk. $F(u, g) = 0$ *so* u *is* C^{k+1}*, and we are using the charts defined by trivializing* TM *over* u *since we only need* C^k*-Banach mfd structures (*F *is only* C^k *anyway). If you want to use the* C^{∞}*-Banach space structures, then trivialize over a* smooth u*, and study* $F(v, g) = 0$ *(where* v *is an abbreviation for* $\exp_{u(s)} v(s)$*,* $v \in W_0^{1,2}(u^* D_{\varepsilon} TM)$*) so replace* u*'s by* v*'s except in the defns of* A_1, B*.*

Cor. $\operatorname{coker} D_{(u,g)} F \subset \ker(D_u F_g)^* \cap (\operatorname{im} P)^{\perp}$.

Claim 2 $D_{(u,g)} F$ is surjective.

Proof. If not, then $\exists b \neq 0 \in E_u$:

(1) $(D_u F_g)^* b = 0$
(2) $\langle Df'_g \cdot \vec{g}, b\rangle_{L^2} = 0 \;\; \forall \vec{g}$.

Key trick 1: (1) $\Rightarrow$ b continuous. (Proof in next Lecture)
Key trick 2: $b(s_0) \neq 0$ for some $s_0 \in \mathbb{R}$. Claim: it suffices to define $\vec{g}$ at $u(s_0)$ with

$$g_M((Df'_g)_{u(s)} \cdot (\vec{g})_{u(s)}, b(s)) > 0 \qquad \text{at } s = s_0 \qquad (*)$$

Proof of Claim:

pick any C^k-extension of $(\vec{g})_{u(s_0)}$ to $\vec{g}_x$ defined for $x \in M$ close to $u(s_0)$
$\Rightarrow$ by continuity $(*)$ holds near s_0
globalize $\vec{g}$: multiply $\vec{g}$ by a bump function ($= 0$ away from $u(s_0)$, $= 1$ at $u(s_0)$)
$\Rightarrow$ $(*)$ holds with "$\geq$" for all s, and with "$>$" near s_0
$\Rightarrow$ (2) fails. Contradiction. $\square$

Construction of $\vec{g}$ as in the Claim:
Locally $f'_g = g^{-1}\partial f$, so

$$Df'_g \cdot \vec{g} = (\partial_t|_{t=0}(g + t\vec{g})^{-1})\, \partial f.$$

Now use the usual series trick:

$$\begin{aligned} (g + t\vec{g})^{-1} \;\; &= \;\; [g \cdot (1 + tg^{-1}\vec{g})]^{-1} \\ & \;\;\;\;\;\; (1 + tg^{-1}\vec{g})^{-1} \cdot g^{-1} \qquad \text{(careful with order of matrices!)} \\ & \;\;\;\;\;\; (1 - tg^{-1}\vec{g} + \text{order}\, t^2) \cdot g^{-1} \end{aligned}$$

Hence $\boxed{Df'_g \cdot \vec{g} = -g^{-1} \cdot \vec{g} \cdot g^{-1} \cdot \partial f}$

Now $\partial f \neq 0$ since[4] $u(s_0) \notin \mathrm{Crit}(f)$.

Moreover, $g^{-1} \cdot \vec{g} \cdot g^{-1}$ is an arbitrary[5] symmetric matrix at s_0 by letting $\vec{g}$ vary: indeed to get the symmetric matrix S take $\vec{g} = gSg$.

$\Rightarrow$ $Df'_g \cdot \vec{g}$ is arbitrary at s_0. So in our case, we pick $\vec{g}$ so that $Df'_g \cdot \vec{g} = b(s_0)$. $\square$

[4] u is a $-\nabla f$ trajectory from p to $q \neq p$, so it is non-constant: the unique $-\nabla f$ trajectory passing through a critical point is the constant trajectory at the critical point.
[5] $G \subset \mathrm{Sym}^2(T^*M)$ is an open subset, so $T_g G = T_g \mathrm{Sym}^2(T^*M)$.