LECTURE 14.

PART III, MORSE HOMOLOGY, 2011

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4.13. **Claim 1.**

Aim: $F(u,g)=0 \Rightarrow D_u F_g = \nabla_s \bullet - \nabla_\bullet f_g'(u) : T_u U \to E_{(u,g)}$ is Fred, index $|p|-|q|$.

Rmk. $F(u,g)=0 \Rightarrow u$ is C^k (Lemma 4.10). We will prove the Aim more generally for any C^k-path $u \in U$ (we will not use $F(u,g)=0$ anymore).

$u \in C^k \Rightarrow$ can trivialize u^*TM by parallel transport:
Pick basis $e_1, \ldots, e_m$ of u^*TM with $\nabla_s e_i = 0$

$$V = \sum V^i e_i$$
$$\nabla_s V = \sum \partial_s V^i e_i$$
$$\nabla_V f_g'(u) = \sum V^j \nabla_{e_j} f_g'(u) = \sum (A_s)^i_j V^j e_i \quad \text{(this defines a matrix } A_s\text{)}$$

Cor. *$D_u F_g$ in the basis e_i is:*

$$\boxed{\begin{array}{ccc} W_0^{1,2}(\mathbb{R},\mathbb{R}^m) & \to & L^2(\mathbb{R},\mathbb{R}^m) \\ \begin{pmatrix} V^1 \\ \vdots \\ V^m \end{pmatrix} & \mapsto & (\partial_s + A_s)\cdot \begin{pmatrix} V^1 \\ \vdots \\ V^m \end{pmatrix} \end{array}}$$

Define the adjoint A_s^* by $\boxed{g_M(A_s^* x, y) = g_M(x, A_s y) \ \ \forall x,y \in \mathbb{R}^m}$.

Lemma. *$(\partial_s + A_s)^* b = 0$ for $b \in L^2 \Rightarrow b \in W^{1,2} \overset{Sobolev}{\Rightarrow} b \in C^0$.*

Proof. $0 = \langle (\partial_s + A_s)^* b, \phi\rangle_{1,2} = \langle b, (\partial_s + A_s)\phi\rangle_{L^2} = 0 \ \ \forall \phi \in L^2$
$\Rightarrow \langle b, \partial_s \phi\rangle_{L^2} = -\langle A_s^* b, \phi\rangle_{L^2} \ \forall \phi \in C_c^\infty$
$\Rightarrow$ weak $\partial_s b = A_s^* b \in L^2$
$\Rightarrow b \in W^{1,2}$. □

Cor. *4.12 **Key trick 1** holds, since $(D_{(u,g)}F_g)^* = (\partial_s + A_s)^*$ in the trivialization.*

Rmk. *In general, such Lemmas are proved by the elliptic regularity theorem.*[1]

Rmk. $\boxed{W_0^{1,2}(\mathbb{R},\mathbb{R}^m) = W^{1,2}(\mathbb{R},\mathbb{R}^m)}$ *and similarly for L^2.*

Proof. Given $b \in W^{1,2}$, $\exists b_n \in C^\infty$ with $b_n \to b$ in $W^{1,2}$. Pick bump $\beta_n : \mathbb{R} \to [0,1]$, $\beta_n = 1$ on $[-n+1, n-1]$ and $\beta_n = 0$ outside $[-n,n]$, and $|\partial_s \beta_n| \le 2$. Then $\beta_n b_n \to b$ in L^2 since $\int_{\mathbb{R}\setminus[-n,n]} |b|^2\, ds \to 0$ as $n \to \infty$. Also, $\partial_s(\beta_n b_n) = (\partial_s \beta_n)\cdot b_n + \beta_n \cdot \partial_s b_n$: the second term $\to \partial_s b$ in L^2 by the same argument, and the first term is supported outside $[-n,n]$ and bounded by $2\int_{\mathbb{R}\setminus[-n,n]} |b|^2\, ds \to 0$ as $n \to \infty$. □

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[1]This is a general bootstrapping result for $W^{k,2}$ spaces. *Non-examinable:* for an elliptic operator L (such as ∂_s or the Laplacian) the weak solutions u of $Lu = f$ with $f \in W^{k,2}$ (and $u = 0$ on ∂U, with smooth ∂U) lie in $u \in W^{k+\text{order}(L),2}$. For $f \in C^\infty = \cap_k W^{k,2}$ get $u \in \cap_k W^{k,2} = C^\infty$.

Rmk. *The same arguments show* $W^{k,p}(\mathbb{R},\mathbb{R}^m) = W_0^{k,p}(\mathbb{R},\mathbb{R}^m)$*. For other examples, see Adams,* Sobolev spaces*, p.70.*

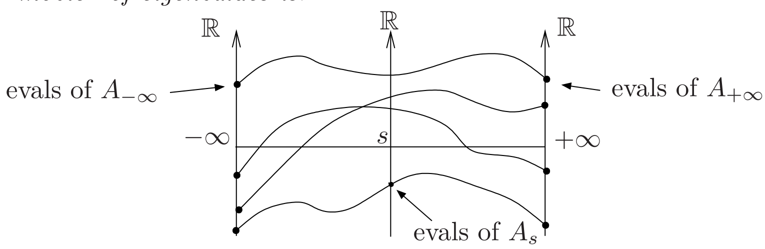
Claim 1 follows immediately from the following theorem:

Thm.

(1) $s \mapsto A_s$ *is a* C^{k-1}*-path of symmetric matrices, with* $A_s \to \mathrm{Hess}_p f$, $\mathrm{Hess}_q f$ *in* C^{k-1} *as* $s \to -\infty, +\infty$*.* Proof. Hwk 2, (1iii). $\square$

(2) $\partial_s + A_s : W^{1,2}(\mathbb{R},\mathbb{R}^m) \to L^2(\mathbb{R},\mathbb{R}^m)$ *is a Fredholm operator*

(3) $\mathrm{index}\,(\partial_s + A_s) =$ *"spectral flow of* A_s*"* $= |p| - |q|$.
Explanation: suppose you could[2] *diagonalize* A_s *continuously in* s*, then the "movie" of eigenvalues is:*



$$\begin{aligned}
\text{spectral flow} &= \#(\textit{evalues of } A_s \textit{ moving from negative to positive}) - \\
&\quad -\#(\textit{evalues of } A_s \textit{ moving from positive to negative}) \\
&= \mathrm{index}\,(\mathrm{Hess}_p f) - \mathrm{index}\,(\mathrm{Hess}_q f) \\
&= |p| - |q|
\end{aligned}$$

In the picture: spectral flow $= 1 - 0$ *and* $|p| - |q| = 3 - 2$

Rmk. *The spectral flow is a useful "relative index" in the* ∞*-dimensional settings when the absolute indices* $|p|, |q|$ *are infinite.*

4.14. $\partial_s + A_s$ **is Fredholm.**

$(*)$ $L = \partial_s + A_s : W^{1,2}(\mathbb{R},\mathbb{R}^m) \to L^2(\mathbb{R},\mathbb{R}^m)$ with $A_s \to A_{\pm\infty}$ symm nondeg
$K = \text{restriction} : W^{1,2}(\mathbb{R},\mathbb{R}^m) \to L^2([-S,S],\mathbb{R}^m)$ compact (by Sobolev)

Aim: deduce $\partial_s - A_s$ has finite dimensional kernel and closed image from Lemma:

Closed range Lemma

A, B, C Banach spaces
$L : A \to B$ bounded linear
$K : A \to C$ compact[3] bounded linear

$$\boxed{\begin{array}{l} \text{If } \ \|a\|_A \le \text{constant} \cdot (\|La\|_B + \|Ka\|_C) \quad \forall a \in A \\ \text{then } \begin{cases} \ker L \text{ finite dimensional} \\ \mathrm{im}\, L \text{ closed} \end{cases} \end{array}}$$

Proof. WLOG constant $= 1$ (rescale L, K)
$\ker L \subset \{a \in A : \|a\|_A \le \|Ka\|_C\}$ closed
Suppose $a_n \in \ker L : \|a_n\| = 1$
$\Rightarrow$ subseq $Ka_n \to c$ (K compact)

[2]This is presumably true generically over $\mathbb{C}$, but we will take a different approach.
[3]This means: if a_n is bdd, then Ka_n has a cgt subseq.

$\Rightarrow \|a_n - a_m\|_A \leq \|Ka_n - Ka_m\|_C \to \|c - c\|_C = 0$
$\Rightarrow a_n$ Cauchy, hence cgt
$\Rightarrow$ unit sphere in $\ker L$ is compact
$\Rightarrow \boxed{\ker L \text{ finite dimensional}}$
$\Rightarrow \exists$ closed complement A_0 of $\ker L$, so $A = \ker L \oplus A_0$ (see 2.3(1))
$\Rightarrow L : A' \to B$ injective.

Suppose $La_n \to b$. WLOG $a_n \in A_0$ (without changing La_n).
Suppose by contradiction that $\|a_n\|$ is unbounded.
$\Rightarrow \exists$ subsequence $\|a_n\| \to \infty$
$\Rightarrow \widetilde{a_n} = \frac{a_n}{\|a_n\|}$ has $L\widetilde{a_n}$ cgt (since $La_n \to b$)
$\Rightarrow \|\widetilde{a_n} - \widetilde{a_m}\|_A \leq \|L\widetilde{a_n} - L\widetilde{a_m}\|_B + \|K\widetilde{a_n} - K\widetilde{a_m}\|_C \to 0$
$\Rightarrow$ Cauchy, hence $\widetilde{a_n} \to \widetilde{a} \in A_0$ (A_0 closed)
$\Rightarrow L\widetilde{a} \to 0 = L\widetilde{a}$
$\Rightarrow \widetilde{a} = 0$ (L injective on A_0)
$\Rightarrow$ contradiction ($\|\widetilde{a_n}\| = 1 \to \|\widetilde{a}\| = 1$ not 0)

$\Rightarrow \boxed{\|a_n\| \text{ bounded}}$.
$\Rightarrow$ subsequence Ka_n cgt
$\Rightarrow \|a_n - a_m\|_A \leq \|La_n - La_m\|_B + \|Ka_n - Ka_m\|_C \to 0$
$\Rightarrow a_n$ Cauchy, so cgt, so $a_n \to a, La_n \to b = La$
$\Rightarrow \operatorname{im} L$ closed. □

Thm. *Inequality of Lemma holds for operators in* $(*)$ *for* $S \gg 0$.

Proof.
Step (1). For $A_s \equiv A_{-\infty}$ constant (symmetric nondegenerate):
Aim: Solve $(\partial_s + A)V = W \in L^2(\mathbb{R}, \mathbb{R}^m)$ then bound V in terms of W $(= LV)$.

$\mathbb{R}^m = E^- \oplus E^+$ where $E^- = \oplus$eigenspaces of $A_{-\infty}$ for evals < 0, $E^+ = \oplus \cdots > 0$
WLOG $\mathbb{R}^m = E^-$ by solving equation separately on E^-, E^+. Solution:[4]

$$V(s) = \textstyle\int_{-\infty}^{s} e^{-A(s-t)} W(t)\, dt$$

Check: $\partial_s V = e^{-A(s-t)} W|_{t=s} - \int_{-\infty}^{s} A e^{-A(s-t)} W\, dt = W(s) - AV(s)$ ✓

Unique solution? suppose $(\partial_s + A)V = 0$
$\Rightarrow$ change coords so that A diagonal
$\Rightarrow \partial_s V^i = -\lambda_i V^i$ $\quad \lambda_i$ evalue
$\Rightarrow V^i(s) = \text{constant} \cdot e^{-\lambda_i \cdot s}$
$\Rightarrow$ not in L^2 unless $V \equiv 0$. ✓

Is V in L^2?

$$\begin{aligned} V(s) &= \textstyle\int_{-\infty}^{\infty} \phi(s-t) W(t)\, dt \qquad \phi(s) = \begin{cases} e^{-As} & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases} \\ &= (\phi * W)(s) \end{aligned}$$

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[4] *Non-examinable:* In general, to solve the ODE $\partial_s V = BV + W$, you first solve for the fundamental solution $\phi : \mathbb{R} \to \operatorname{End}(\mathbb{R}^m)$ of $\partial_s \phi = B\phi$, $\phi(s_0) = I$ (so that $V(s) = \phi(s)V_0$ solves $\partial_s V = BV$ with $V(s_0) = V_0$), then our ODE has solution

$$V(s) = \phi(s)V_0 + \phi(s) \textstyle\int_{s_0}^{s} \phi(t)^{-1} W(t)\, dt = \phi(s)V_0 + \int_{s_0}^{s} \phi(s-t) W(t)\, dt$$

as can be checked by differentiating. The first piece is the homogeneous solution, the second is the particular solution. In our particular example, $\phi(s) = e^{-A \cdot s}$, and "$V(-\infty)$" $= 0$.

$$\begin{array}{rcll}
\|\phi * W\|_2^2 & = & \int_{-\infty}^{\infty} |\int_{-\infty}^{\infty} \phi(s-t)W(t)\,dt|^2\,ds & \\
& \leq & \int (\int |\phi(s-t)|^{1/2} \cdot |\phi(s-t)|^{1/2} \cdot |W(t)|\,dt)^2\,ds & \\
& \leq & \int (\int |\phi(s-t)|\,dt \cdot \int |\phi(s-t)| \cdot |W(t)|^2\,dt)\,ds & \text{(Cauchy-Schwarz)} \\
& = & \|\phi\|_1 \iint |\phi(s-t)| \cdot |W(t)|^2\,ds\,dt & \text{(Fubini)} \\
& = & \|\phi\|_1 \cdot \|\phi\|_1 \cdot \|W\|_2^2 &
\end{array}$$

$\Rightarrow \boxed{\|V\|_2 = \|\phi * W\|_2 \leq \|\phi\|_1 \cdot \|W\|_2}$ (a version of Young's inequality)

<u>Is V in $W^{1,2}$?</u>

$\partial_s V = -AV + W$

$$\begin{array}{rcl}
\|\partial_s V\|_2 & \leq & \|AV\|_2 + \|W\|_2 \\
& \leq & \|A\| \cdot (\|\phi\|_1 + 1) \cdot \|W\|_2
\end{array}$$

where $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$. Thus $\|V\|_2 + \|\partial_s V\|_2 \leq \text{constant} \cdot \|W\|_2$, equivalently:

$$\boxed{\|V\|_{1,2} \leq \text{constant} \cdot \|(\partial_s + A_{-\infty})V\|_2}$$

Step (2). For $V = 0$ away from $-\infty$:

$\|A_s - A_{-\infty}\| \leq \varepsilon$ for $s < -S + 1$ (for large S depending on ε)

Assume $V = 0$ for $s \geq -S + 1$:

$$\begin{array}{rcll}
\Rightarrow \|V\|_{1,2} & \leq & c \cdot \|(\partial_s + A_{-\infty})V\|_2 & \text{(by (1))} \\
& \leq & c \cdot (\|(\partial_s + A_s)V\|_2 + \|(A_s - A_{-\infty})V\|_2) & \text{(triangle ineq)} \\
& \leq & c \cdot (\|LV\|_2 + \varepsilon\|V\|_2) & (L = \partial_s + A_s) \\
\Rightarrow \|V\|_{1,2} & \leq & \frac{c}{1 - c\varepsilon}\|LV\|_2 & \text{(pick } \varepsilon > 0 \text{ small so } c\varepsilon < 1)
\end{array}$$

Step (3). Similarly do (1) for $A_{+\infty}$, and (2) for $V = 0$ for $s \leq S - 1$.

Step (4). General case:

Pick a bump function $\beta \in C^\infty(\mathbb{R}, [0,1])$ with $\beta = \begin{cases} 1 & \text{for } |s| \leq S - 1 \\ 0 & \text{for } |s| > S \end{cases}$

$$\begin{array}{rcll}
\|(1-\beta)V\|_{1,2} & \leq & \text{constant} \cdot \|L(1-\beta)V\|_2 & \text{(by (2), (3))} \\
\|\beta V\|_{1,2} & \equiv & \|\beta V\|_2 + \|\partial_s(\beta V)\|_2 & \\
& \leq & \|\beta V\|_2 + \|L\beta V\|_2 + (\sup_s \|A_s\|) \cdot \|\beta V\|_2 & \\
& \leq & \text{constant} \cdot (\|\beta V\|_2 + \|L\beta V\|_2) &
\end{array}$$

$$\begin{array}{rcll}
\|V\|_{1,2} & \leq & \|(1-\beta)V\|_{1,2} + \|\beta V\|_{1,2} & \text{(triangle ineq)} \\
& \leq & \text{constant} \cdot (\|\beta V\|_2 + \|L\beta V\|_2 + \|L(1-\beta)V\|_2) & \text{(above ineq's)} \\
& \leq & \text{constant} \cdot (\|KV\|_2 + \|LV\|_2) & \text{(see below)}
\end{array}$$

The last inequality used that $\beta, \partial_s\beta$ are bounded and vanish outside $[-S, S]$, so

$$\|(\partial_s\beta)V\|_2 \leq \text{constant} \cdot \|V|_{[-S,S]}\|_2 \equiv \text{constant} \cdot \|KV\|_2,$$

and it used that $\|\beta LV\|_2 \leq \text{constant} \cdot \|LV\|_2$ and similarly for $\|(1-\beta)LV\|_2$. □

Cor. *$L = \partial_s + A_s$ has closed image and finite dimensional kernel.*

Aim of next lecture: the cokernel is also finite dimensional, so $\partial_s + A_s$ is Fred.