LECTURE 15.

PART III, MORSE HOMOLOGY, 2011

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4.14. $\partial_s + A_s$ Fredholm (continued).

$\boxed{L = \partial_s + A_s : W^{1,2}(\mathbb{R}, \mathbb{R}^m) \to L^2(\mathbb{R}, \mathbb{R}^m)}$

Aim: L has finite dimensional cokernel.

Lemmas 4.11 $\Rightarrow$ $\operatorname{coker} L \cong (\operatorname{im} L)^\perp \cong \ker L^*$.
Warning. $L^* : L^2 \to W^{1,2}$ is difficult to calculate (try!).[1]
Remedy: the *formal adjoint* $L^\star$, which is not L^*, but it has a similar kernel.

Def. *$X, Y \to M$ vector bundles with metrics over a (possibly non-compact) Riem mfd. Given a linear map on sections $L : C^\infty(X) \to C^\infty(Y)$, the* formal adjoint *(if it exists) is a map $L^\star : C^\infty(Y) \to C^\infty(X)$ satisfying*

$$\int_M \langle Lx, y\rangle_Y \, vol_M = \int_M \langle x, L^\star y\rangle_X \, vol_M \qquad \forall x \in C_c^\infty(X), \forall y \in C_c^\infty(Y).$$

Exercise: $L^\star$ is unique if it exists.

Lemma. *For $L = \partial_s + A_s$, get*[2] *$L^\star = -\partial_s + A_s^*$.*

Proof. $X = Y = \mathbb{R}^m$ and we trivialize u^*TM by g_M-ortonormal[3] e_i. Therefore $\langle \cdot, \cdot\rangle_X = \langle \cdot, \cdot\rangle_Y = \langle \cdot, \cdot\rangle_{\mathbb{R}^m}$. For all $x, y \in C_c^\infty(\mathbb{R}, \mathbb{R}^m)$, by definition of the weak derivative:[4]

$$\int \langle (\partial_s + A_s)x, y\rangle \, ds = \int \langle x, -\partial_s y\rangle \, ds + \int \langle x, A_s^* y\rangle \, ds. \quad \square$$

Cor. *$L^\star$ extends to $-\partial_s + A_s^* : W^{1,2} \to L^2$, so $\langle L^\star x, y\rangle_2 = \langle x, Ly\rangle_2, \forall x, y \in W^{1,2}$.*

Proof. approximate x, y by C_c^∞. $\square$

Cor. $\operatorname{coker} L \cong \ker L^\star = \ker(-\partial_s + A_s^*)$

Proof. $\operatorname{coker} L \cong (\operatorname{im} L)^\perp \overset{4.11}{\cong} \ker L^* \subset W^{1,2}(\mathbb{R}, \mathbb{R}^m)$ (Lemma 4.13).[5]
$\Rightarrow$ if $b \in (\operatorname{im} L)^\perp$ then $0 = \langle La, b\rangle_2 = \langle a, L^\star b\rangle_2$ for all a by the previous Cor, so $b \in \ker L^\star$, and conversely if $b \in \ker L^\star$ then $b \in (\operatorname{im} L)^\perp$ by the same equation.
$\Rightarrow \ker L^\star \cong (\operatorname{im} L)^\perp \cong \operatorname{coker} L$. $\square$

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[1] The issue is that it involves $\langle x, L^* y\rangle_{W^{1,2}}$ not just L^2 pairings.

[2] Where $g_M(A_s v, w) = g_M(v, A_s^* w), \forall v, w$. In our case A_s is symmetric, so $A_s = A_s^*$.

[3] Pick e_i orthonormal at a point, then parallel transport will preserve orthonormality:
$\partial_s \langle e_i, e_j\rangle = \langle \nabla_s e_i, e_j\rangle + \langle e_i, \nabla_s e_j\rangle = 0$ since $\nabla_s e_i = 0$.

[4] in this case, we are really just doing integration by parts.

[5] Observe that we are first checking that $b \in L^2$ really is $W^{1,2}$ before we apply $L^\star$!

4.15. **Index of $\partial_s + A_s$.**

Cor. $\boxed{\mathrm{Index}(L) = \dim\ker L - \dim\ker L^\star}$

Aim: find $\ker L, \ker L^\star$ to calculate dimensions.
Need to solve
$$\partial_s x = -A_s x \qquad x : \mathbb{R} \to \mathbb{R}^m \qquad x(s_0) = x_0$$
write $x(s) = \phi(s)\cdot x_0$, for $\phi : \mathbb{R} \to \mathrm{End}(\mathbb{R}^m) = m \times m$ matrices. So need to solve:
$$\partial_s \phi = -A_s \cdot \phi, \qquad \phi(s_0) = I \qquad (\text{so } x(s_0) = \phi(s_0)x_0 = x_0)$$
This ϕ is called **fundamental solution** (and later we will write $\phi(s) = \phi^s_{s_0}$ to emphasize the initial condition).
By 2.1 Rmk 2 $\Rightarrow \exists$ unique $\phi \in C^{k-1}$ since $A : \mathbb{R} \to \mathrm{End}\,\mathbb{R}^m$ is C^{k-1}-bounded.[6]

Consider $\boxed{E^\pm = E^\pm(s_0) = \{x_0 \in \mathbb{R}^m : \phi(s)x_0 \to 0 \text{ as } s \to \pm\infty\}}$

These are vector subspaces of $\mathbb{R}^m$ since the ODE is linear.

Recall $L = \partial_s + A_s$, $L^\star = -\partial_s + A_s^*$, where $A_{\pm\infty}$ are symmetric non-singular.

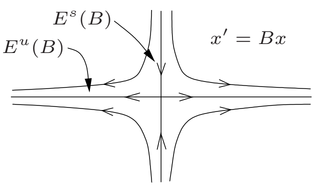
Thm.
$\ker L \cong E^- \cap E^+$,
$\ker L^\star \cong (E^-)^\perp \cap (E^+)^\perp$,
$E^- \cong E^u(-A_{-\infty}) = E^s(A_{-\infty}) \equiv \oplus$*eigenspaces for evals* $\lambda < 0$ *of* $A_{-\infty} = \mathrm{Hess}_p f$,
$E^+ \cong E^s(-A_{+\infty}) = E^u(A_{+\infty}) \equiv \oplus$*eigenspaces for evals* $\lambda > 0$ *of* $A_{+\infty} = \mathrm{Hess}_q f$.

Cor.
$$\begin{aligned}\mathrm{Index}\, L &= \dim E^- \cap E^+ - \dim((E^-)^\perp \cap (E^+)^\perp) \\ &= \dim E^- \cap E^+ - (m - \dim(E^- + E^+)) \quad (\textit{taking } \perp \textit{ of } \cap \textit{ gives } +) \\ &= \dim E^- + \dim E^+ - m \\ &= |p| - |q|.\end{aligned}$$

4.16. **Dynamical systems and ODEs.**
Aim: study $x'(t) = A(t)\cdot x(t)$, $A(t)$ matrices converging to a nonsingular matrix B as $t \to +\infty$. Want to compare solutions with the solutions of $x'(t) = B \cdot x(t)$.

Rmk. *We use a time variable $t \in \mathbb{R}$ instead of $s \in \mathbb{R}$ to emphasize that this is an abstract problem separate from previous sections.*

Example: $B = \left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$. Then
$$\begin{aligned}E^u(B) &= \{x_0 : \mathrm{flow}(x_0) \to 0 \text{ as } t \to -\infty\} = \mathbb{R}\cdot\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\\ E^s(B) &= \{x_0 : \mathrm{flow}(x_0) \to 0 \text{ as } t \to +\infty\} = \mathbb{R}\cdot\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\end{aligned}$$



$$\begin{aligned}x &= \exp(Bt)\cdot x \\ &= \begin{bmatrix} e^t & 0 \\ 0 & e^{-t}\end{bmatrix} x_0\end{aligned}$$

Now look at a flowline for $x'(t) = A(t)x(t)$ starting at x_0 for $t = t_0$:

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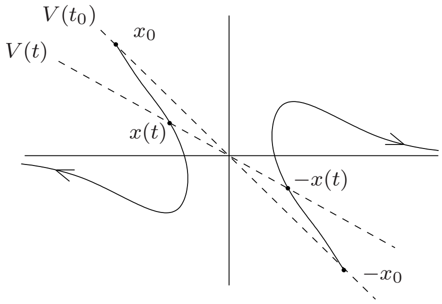
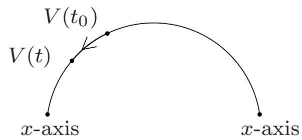
[6] $s \mapsto A_s$ is C^{k-1}, $A_s \to A_{\pm\infty}$ in C^{k-1} as $s \to \pm\infty$, so for $s \gg 0$: $\|A_s\| \approx \|A_{\pm\infty}\|$, $\|\partial_s^j A_s\| \approx 0$.



Because the ODE is linear, we can in fact consider the flow of the lines $V(t) = \phi^t_{t_0} \cdot (\mathbb{R} \cdot x_0)$, which viewed as a flow on the Grassmannian $\mathbb{R}P^1 = \{\text{lines in } \mathbb{R}^2\}$ is:



which shows that $V(t) \to E^u(B) = x$-axis as $t \to +\infty$, so $E^u(B)$ is an attractor.

Aim of next Lecture: prove in general that $E^u(B)$ is an attractor in the Grassmannian of k-planes, $k = \dim E^u(B)$, for the flow $x'(t) = A(t)x(t)$ (assuming B is nonsingular and symmetric and $A(t) \to B$ as $t \to +\infty$).