# LECTURE 16.

PART III, MORSE HOMOLOGY, 2011

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4.16. **Dynamical systems and ODEs (continued).**

**Def.** *$B$ is a* hyperbolic matrix *if the real parts $Re$(eigenvalues) $\neq 0$.*

**Rmk.** *These matrices are generic. Exercise: check the details of this lecture for $B$ symmetric and nondegenerate (so the eigenvalues are real and non-zero, so it's hyperbolic), indeed just assume WLOG that is $B$ diagonal (by a change of basis).*

**Linear Algebra Trick**[1]

*For $L : \mathbb{R}^n \to \mathbb{R}^n$ linear with evalues $\lambda$ satisfying*

$$\boxed{a < \operatorname{Re}\lambda < b}$$

*there exists a basis of $\mathbb{R}^n$ inducing*[2] *an inner product $(\cdot,\cdot)$ with norm $(\!(\cdot)\!)$ such that*

$$a(\!(x)\!)^2 \leq (Lx, x) \leq b(\!(x)\!)^2.$$

**Key trick.** If $x'(t) = Lx(t)$ then

$$\frac{d}{dt}(\!(x)\!)^2 = \frac{d}{dt}(x,x) = 2(Lx,x)$$

so $a \leq \frac{1}{2}\frac{d}{dt}\log(\!(x)\!)^2 = \frac{d}{dt}\log(\!(x)\!) \leq b$ so $\boxed{e^{at}(\!(x(0))\!) \leq (\!(x(t))\!) \leq e^{bt}(\!(x(0))\!)}$.

**Example.** For $L$ restricted to[3] $\bigoplus_{\operatorname{Re}\lambda<0} E_\lambda(L)$ can pick $b < 0$ so $(\!(x(t))\!) \to 0$ exp fast.

**Def.** *Recall $e^{Bt}$ is the flow for $x' = Bx$. The stable and unstable spaces for $B$:*

$$E^s(B) = \{x_0 \in \mathbb{R}^m : e^{Bt}\cdot x_0 \to 0 \ \textit{as}\ t \to +\infty\}$$
$$E^u(B) = \{x_0 \in \mathbb{R}^m : e^{Bt}\cdot x_0 \to 0 \ \textit{as}\ t \to -\infty\}$$

**Rmk.** $\boxed{E^s(-B) = E^u(B)}$ *since* $\lim_{t\to+\infty} e^{-Bt}\cdot x_0 = 0 \Leftrightarrow \lim_{t\to-\infty} e^{Bt}\cdot x_0 = 0$.

**Exercise.**[4] $B$ hyperbolic $\Rightarrow$
$$\boxed{\begin{array}{rcl} E^s(B) & = & \bigoplus_{\operatorname{Re}\lambda<0} E_\lambda(B) \\ E^u(B) & = & \bigoplus_{\operatorname{Re}\lambda>0} E_\lambda(B) \\ E^s(-B^*) & = & E^s(B)^\perp \end{array}}$$

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*Date*: May 2, 2011, © Alexander F. Ritter, Trinity College, Cambridge University.

[1]For a proof, see the 1974 edition of Hirsch-Smale, *Differential Equations*, p.147. For diagonal matrices this is obvious, so for symmetric matrices it's easy (take an orthonormal basis of evectors). You need to fiddle around with the Jordan Normal Form in the general case.

[2]$(x,y) = \sum x_i y_i$ where $x_i, y_i$ are the coordinates of $x, y$ in the given basis, and $(\!(x)\!) = (x,x)^{1/2}$. We reserve the symbols $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, $\perp$ for the standard $\mathbb{R}^n$ symbols.

[3]$E_\lambda(L) = \ker(L - \lambda I)^n$ is the generalized eigenspace for $\lambda$.

[4]For $B$ symmetric, $E^s(-B^*) = E^s(-B) = E^u(B)$.

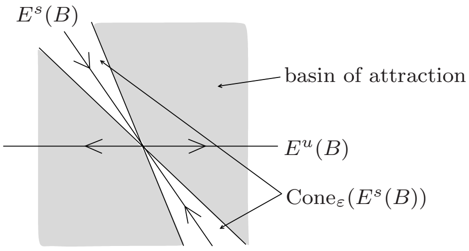
**Thm.** *Suppose $B$ hyperbolic and $A(t) \to B$ as $t \to +\infty$. Then for $t \gg 0$, $E^u(B)$ is an attractor*[5] *inside*

$$\boxed{\mathrm{Gr} = \{k\text{-planes in } \mathbb{R}^m\}}$$

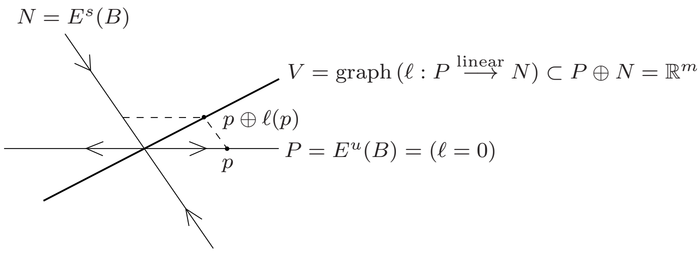
*for the flow $x'(t) = A(t)x(t)$ (where $k = \dim E^u(B)$), with basin of attraction*[6]

$$\mathrm{Gr} \setminus (k\text{-planes intersecting } \mathrm{Cone}_\varepsilon(E^s(B))$$

*where*[7] *$\varepsilon > 0$ can be chosen arbitrarily small for $t \gg 0$.*




*Proof.* Define local coordinates on Gr:




The $(m-k)\cdot k$ local coordinates are the entries of the matrix $\ell$, and these coordinates are defined on $\mathrm{Gr} \setminus (k\text{-planes } V \text{ intersecting } N)$. We abbreviated $E^u(B), E^s(B)$ as $P, N$ since they involve evalues with, respectively, positive and negative real parts.

$$\begin{array}{rcl} V = \ \mathrm{span}\,\{p_i \oplus \ell(p_i)\} & \overset{e^{Bt}}{\mapsto} & e^{Bt} \cdot V = \mathrm{span}\,\{e^{Bt}p_i \oplus e^{Bt} \cdot \ell(p_i)\} \\ \mathrm{graph}(\ell : p_i \mapsto \ell(p_i)) & \mapsto & \mathrm{graph}(e^{Bt} \cdot \ell \cdot e^{-Bt} : e^{Bt}p_i \mapsto e^{Bt} \cdot \ell(p_i)) \end{array}$$

So the local ODE is:

$$\ell' = \left.\frac{d}{dt}\right|_{t=0} e^{Bt}\ell e^{-Bt} = B\ell - \ell B,$$

so $\ell' = L\ell$ for the linear map $L : \ell \mapsto B\ell - \ell B$ for[8]

$$\ell \in \mathrm{Lin}(P, N) \cong \mathrm{Lin}(\mathbb{R}^k, \mathbb{R}^{m-k}) \cong \mathbb{R}^{m-k} \otimes (\mathbb{R}^k)^*.$$

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[5] meaning: nearby solutions converge to $E^u(B)$ as $t \to +\infty$.

[6] the subset which specifies what "nearby" means in the previous footnote.

[7] Recall $\mathbb{R}^m = E^s(B) \oplus E^s(-B^*)$ (orthogonal subspaces), so we uniquely decompose $x = x^s + x^u$ in the respective subspaces with $x^s \perp x^u$. Easy check: $\|x\|^2 = \|x^s\|^2 + \|x^u\|^2$. Define

$$\boxed{\mathrm{Cone}_\varepsilon(E^s(B)) = \{x \in \mathbb{R}^m : \|x^u\|^2 < \varepsilon \|x^s\|^2\}}$$

[8] The last isomorphism works as follows.
Identify $\mathbb{R}^k \cong (\mathbb{R}^k)^* \equiv \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R})$ by $v \mapsto (v^T : x \mapsto v^T x = \langle v, x\rangle \in \mathbb{R})$. The linear map $w \otimes v^T$ is $\mathbb{R}^k \to \mathbb{R}^{m-k}, x \mapsto w \cdot (v^T x)$. If $p_i$ are an orthonormal basis, then $\ell = \sum \ell(p_i) \otimes p_i^T$.

We want to find the eigenvalues of $L$. Observe

$$L(w \otimes v^T) = Bw \otimes v^T - w \otimes v^T B = Bw \otimes v^T - w \otimes (B^T v)^T.$$

So if $w \in N$ evector of $B$ for evalue $\lambda_w$, and $v \in P$ evector for $B^T$ for evalue $\mu_v$,

$$L(w \otimes v^T) = (\lambda_w - \mu_v)\ \ w \otimes v^T,$$

so $w \otimes v^T$ is an evector of $L$ for evalue $\lambda_w - \mu_v$.

**Exercise.** the multiplicities (dim of gen. espaces) are what you expect.

$B$ preserves $P, N$, and evalues of $B|_P$ = evalues of $B^T|_P$, so

$$\operatorname{Re}\lambda_w < 0,\ \operatorname{Re}\mu_v > 0$$

so $\operatorname{Re}(\lambda_w - \mu_v) < 0$. So by the Key trick applied to $L$,

$$(L\ell, \ell) \leq -\delta (\!(\ell)\!)^2 < 0 \qquad (*)$$

for any $-\delta <$ smallest Real part of all such differences of eigenvalues $\lambda_w - \mu_v$.

Now consider $x' = A(t)x$. Let $\phi^t_{t_0}$ denote its flow,[9] so

$$\ell(t) = \phi^t_{t_0}(\ell(t_0)) = \text{flow of } \ell(t_0)$$

*Warning: the flow can be unbounded since the flow of $V$ may intersect $E^s(B)$, so $\ell \to \infty$, i.e. you exit the chart for Gr!*

For $t_0 \gg 0$, and away from a cone around $E^s$:

$$\left.\frac{d}{dt}\right|_{t=t_0} \phi^t_{t_0}(\ell(t_0)) \approx L \cdot \ell(t_0) \qquad (**)$$

since $A(t_0) \to B$ as $t_0 \to +\infty$.

$$\begin{aligned}
\Rightarrow \tfrac{d}{dt}(\ell(t), \ell(t)) &= 2(\ell'(t), \ell(t)) \\
&= 2(\left.\tfrac{d}{ds}\right|_{s=t} \ell(s), \ell(t)) \\
&= 2(\left.\tfrac{d}{ds}\right|_{s=t} \phi^s_t(\ell(t)), \ell(t)) \\
&\leq 2(-\tfrac{\delta}{2}(\!(\ell)\!)^2)
\end{aligned}$$

where in the last line we combined $(**)$ and $(*)$. So arguing as in the Key trick,

$$(\!(\ell(t))\!) \leq e^{-\frac{\delta}{2}(t-t_0)}(\!(\ell(t_0))\!) \to 0 \ \ (\text{“} = E^u(B)\text{”})\ \text{exponentially fast.} \quad \square$$

**Aim**: $E^s(t_0)$ converges to $E^s(B)$ as $t_0 \to \infty$. The idea is: solutions of $x'(t) = A(t)x(t)$ which decay to zero as $t \to \infty$ cannot get attracted to $E^u(B)$ because if you are close to $E^u(B)$ then you flow off to infinity. The difficulty is that we don't know the dimension of $E^s(t_0)$, and a priori the dimension can jump at $t_0 = \infty$ since we haven't proved a continuity “at infinity” result. The key idea is: show that the adjoint problem produces a space orthogonal to $E^s(t_0)$, and use that cleverly.

**Def.** *Write $\phi^t_{t_0}$ for the flow for $x' = A(t)x$ starting at $t_0$, and $\widetilde{\phi}^t_{t_0}$ for $x' = -A(t)^* x$.*

$$\begin{aligned}
E^s(t_0) &= \{x_0 \in \mathbb{R}^m : \phi^t_{t_0}(x_0) \to 0 \ \textit{as}\ t \to +\infty\} \\
\widetilde{E}^s(t_0) &= \{x_0 \in \mathbb{R}^m : \widetilde{\phi}^t_{t_0}(x_0) \to 0 \ \textit{as}\ t \to +\infty\}
\end{aligned}$$

**Lemma.**

$$\boxed{\mathbb{R}^m = E^s(t_0) \oplus^{\perp} \widetilde{E}^s(t_0) \to E^s(B) \oplus^{\perp} E^s(-B^*) \ \textit{as}\ t_0 \to +\infty}$$

[9]fundamental solution with initial condition $t_0$, so $\partial_t \phi^t_{t_0} = A(t) \circ \phi^t_{t_0}$ and $\phi^{t_0}_{t_0} = I$.

*Proof.*

***Claim 1:*** $E^s(t_0)$ gets "attracted" to $E^s(B)$, meaning: $E^s(t_0)$ lies in an arbitrarily small cone around $E^s(B)$ for large $t_0$ (not claiming $\dim E^s(t_0) = \dim E^s(B)$ yet).

*Proof:* Suppose not. Then some vector $x \in E^s(t_0)$ lies outside that small cone. So $x$ lies in the basin of attraction of $E^u(B)$ by the Theorem.[10] But once $x$ stays within a small cone around $E^u(B)$, the flow of the vector $x$ will go off to infinity exponentially fast by Hwk 16, ex.1.[11] ✓

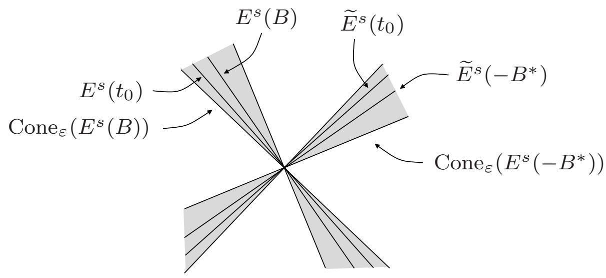
So, via this argument, the Theorem has shown that $E^s(t_0), \widetilde{E}^s(t_0)$ get "attracted" to $E^s(B), E^s(-B^*)$, but we still need to show that the dimensions agree.

Observe: any vector subspace inside $\mathrm{Cone}_\varepsilon(E^s(B))$ has dimension at most $\dim E^s(B)$, by basic linear algebra.[12] So $\boxed{\dim E^s(t_0) \leq \dim E^s(B)}$, similarly for $\widetilde{E}$.

**Key observation:**

$$\partial_t \langle \phi^t_{t_0}(x_0), \widetilde{\phi}^t_{t_0}(y_0)\rangle = \langle A(t)\phi^t_{t_0}(x_0), \widetilde{\phi}^t_{t_0}(y_0)\rangle + \langle \phi^t_{t_0}(x_0), -A(t)^*\widetilde{\phi}^t_{t_0}(y_0)\rangle = 0$$

so $\langle \phi^t_{t_0}(x_0), \widetilde{\phi}^t_{t_0}(y_0)\rangle$ is constant. For $x_0 \in E^s(t_0), y_0 \in \widetilde{E}^s(t_0)$ this constant is zero[13] so at $t = t_0$: $\langle x_0, y_0\rangle = 0$. So $\boxed{E^s(t_0) \perp \widetilde{E}^s(t_0)}$.




***Claim 2:*** $E^s(t_0) = \widetilde{E}^s(t_0)^\perp \cap \mathrm{Cone}_\varepsilon(E^s(B))$ (in particular $\dim E^s(t_0) = \dim E^s(B)$).

*Proof.* Suppose not, by contradiction.

$\Rightarrow \exists x_0 \in \widetilde{E}^s(t_0)^\perp \cap \mathrm{Cone}_\varepsilon(E^s(B))$ but $x_0 \notin E^s(t_0)$

$\Rightarrow x(t) = \phi^t_{t_0}(x_0) \in \widetilde{E}^s(t)^\perp = (\widetilde{\phi}^t_{t_0}\widetilde{E}^s(t_0))^\perp$ (by the Key observation)

$\widetilde{E}^s(t) \subset \mathrm{Cone}_\varepsilon(E^s(-B^*)),\ \forall t \gg 0$ (by Claim 1)

$\Rightarrow x(t) \in (\mathrm{Cone}_\varepsilon(E^s(-B^*)))^\perp = \mathrm{Cone}_\varepsilon(E^s(B)),\ \forall t \gg 0$

By Hwk 16: for $t_0 \gg 0$, $x_0 \in \mathrm{Cone}_\varepsilon(E^s(B))$ (with $\varepsilon > 0$ sufficiently small),

$$\frac{d}{dt}(\!(\phi^t_{t_0}(x_0))\!)^2 \leq -\delta(\!(x_0)\!)^2 \quad \text{if } \phi^t_{t_0}(x_0) \in \mathrm{Cone}_\varepsilon(E^s(B))\ \forall t \geq t_0.$$

So, in our case, $(\!(x(t))\!) \leq e^{-\frac{\delta}{2}(t-t_0)}(\!(x_0)\!) \to 0$, so $x_0 \in E^s(t_0)$. Contradiction.

Similarly $\widetilde{E}^s(t_0) = E^s(t_0)^\perp \cap \mathrm{Cone}_\varepsilon(E^s(-B^*))$.

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[10] In order to work on Gr, you just turn $x$ into a subspace of the correct dimension, $k$, by taking the span of $x$ and any $k-1$ vectors from $E^u(B)$.

[11] Strictly speaking Hwk 16 ex.1 proves exponential convergence to 0 of vectors whose flow stays in a small cone around $E^s(B)$, but a similar argument proves what we mentioned here.

[12] The vector subspace in fact lies inside the graph of a linear map constructed as in the local coords construction for Gr.

[13] since both vectors in the inner product converge to zero as $t \to \infty$ by definition.

So $\mathbb{R}^m = E^s(t_0) \oplus \widetilde{E}^s(t_0)$, the summands are $\perp$ to each other, and given any small $\varepsilon > 0$, $E^s(t_0) \oplus \widetilde{E}^s(t_0) \subset \mathrm{Cone}_\varepsilon(E^s(B)) \oplus \mathrm{Cone}_\varepsilon(E^s(-B^*))$ for $t_0 \gg 0$. □

**Cor.** *$E^s(t_0) \to E^s(B)$ as $t_0 \to \infty$ and the dimensions agree.*
*Similarly, if $A(t) \to C$ as $t \to -\infty$ then*

$$E^u(t_0) \equiv \{x_0 \in \mathbb{R}^m : \phi^t_{t_0} x_0 \to 0 \text{ as } t \to -\infty\} \to E^u(C) \text{ as } t_0 \to -\infty.$$

*Proof.* The first claim is by the Lemma.
Note $x'(t) = A(t)x(t) \Rightarrow y(t) = x(-t)$ satisfies $y'(t) = -A(-t)y(t)$.
$\Rightarrow E^u_{A(t)}(t_0) = E^s_{-A(-t)}(-t_0) \to E^s(-C) = E^u(C)$ as $t_0 \to \infty$. □

4.17. **Proof of Theorem 4.15.** We go back to the notation of 4.15:

$$\begin{array}{lcl} Lx = 0 & \Leftrightarrow & x'(s) = -A_s x(s) \\ L^\star x = 0 & \Leftrightarrow & x'(s) = A_s^* x(s) = -(-A_s^*)x(s) \end{array}$$

Notice that we are now using $s, -A_s$ in place of the $t, A(t)$ of Section 4.16.
By Cor 4.16, $E^+ \to E^s(-A_{+\infty}) = E^u(A_{+\infty})$ as $s_0 \to +\infty \Rightarrow 4^{th}$ claim Thm 4.15.✓
By Cor 4.16, $E^- \to E^u(-A_{-\infty})$ as $s_0 \to -\infty \Rightarrow 3^{rd}$ claim Thm 4.15.✓

**Lemma** ($1^{st}$ claim Thm 4.16)**.**

$$\begin{array}{rcl} \ker(\partial_s + A_s) & \cong & E^- \cap E^+ \\ (x : \mathbb{R} \to \mathbb{R}^m) & \to & x(s_0) \\ (x(s) = \phi^s_{s_0} \cdot x_0) & \leftarrow & x_0 \end{array}$$

*where $\phi$ denotes the flow for $x'(s) = -A_s x(s)$.*

*Proof.* Suppose $x \in \ker(\partial_s + A_s : W^{1,2} \to L^2)$. By 4.10, $x$ is $C^k$ and satisfies $x(s) = \phi^s_{s_0} \cdot x(s_0)$. By Sobolev, $x \to 0$ at $\pm\infty$ since $x \in (C^0, \|\cdot\|_\infty)$. So $x(s_0) \in E^-(s_0) \cap E^+(s_0)$. Conversely: for $x_0 \in E^- \cap E^+$ need $x(s) \in W^{1,2}$ so need to prove fast convergence at the ends. By Hwk 17 (*Exponential Convergence at the ends*):

$$(\partial_s + A_s)x(s) = 0,\ x \in C^1 \Rightarrow \begin{cases} \text{either } |x(s)| \to \infty \text{ as } s \to +\infty \text{ or } -\infty, \text{ so } x \notin L^2 \\ \text{or } x(s) \to 0 \text{ exponentially fast as } |s| \to \infty, \text{ so } x \in L^2 \end{cases}$$

So $x_0 \in E^- \cap E^+ \Rightarrow x \to 0 \Rightarrow x \in L^2 \Rightarrow \partial_s x = -A_s x \in L^2 \Rightarrow x \in W^{1,2}$. □

**Cor** ($2^{nd}$ claim Thm 4.16)**.** $\ker(\partial_s - A_s^*) \cong \widetilde{E}^- \cap \widetilde{E}^+ = (E^-)^\perp \cap (E^+)^\perp$.

4.18. **The clever way to calculate the Index.**
**Aim**: if you only care about the index of $\partial_s + A_s$, then one can avoid the difficult asymptotic analysis of 4.16. Trick: the index will not change if homotope $A_s$ so that it becomes constant in $s$ near $s = \pm\infty$.

Consider

$$A_\bullet : \mathbb{R} \to \mathrm{End}(\mathbb{R}^m) \quad C^{k-1}\text{-differentiable}$$

with $A_s \to A_\pm$ as $s \to \pm\infty$, and $A_\pm$ hyperbolic matrices. By 4.14 (and Hwk 15),

$$\partial_s + A_s : X \to Y \text{ is Fredholm} \quad (X = W^{1,2}(\mathbb{R}, \mathbb{R}^m), Y = L^2(\mathbb{R}, \mathbb{R}^m))$$

The Index of a Fred operator is constant under small-operator-norm perturbations (by 2.3). If $\widetilde{A_s}$ is a homotopy of $A_s$:

$$\begin{array}{rcl} \|(\partial_s + \widetilde{A_s})V - (\partial_s + A_s)V\|_2 & = & \|\widetilde{A_s}V - A_s V\|_2 \\ & \le & \sup_{s\in\mathbb{R}} \|\widetilde{A_s} - A_s\| \cdot \|V\|_2 \\ & \le & \sup_{s\in\mathbb{R}} \|\widetilde{A_s} - A_s\| \cdot \|V\|_{1,2} \end{array}$$

So if $\|\widetilde{A_s} - A_s\|$ is small enough for all $s$, the indices of $\partial_s + \widetilde{A_s}$ and $\partial_s + A_s$ agree.

This proves index invariance for small homotopies. To prove index invariance for a general homotopy $(A_{s,\lambda})_{0\leq\lambda\leq 1}$: the above shows that the index of $\partial_s + A_{s,\lambda}$ is constant on the interval $(\lambda - \delta_\lambda, \lambda + \delta_\lambda)$ (some small $\delta_\lambda > 0$). Now cover $[0,1]$ by finitely many such intervals. Hence, the index is constant for all $\lambda \in [0,1]$.

**Upshot:** By homotopying $A_s$, we may assume:

$$A_s = \begin{cases} A_+ & \text{for } s \geq S \\ A_- & \text{for } s \leq -S \end{cases}$$

Then $E^+(s_0) \equiv E^s(-A_+)$ for $s_0 \geq S$ since both definitions involve the same equation $x'(s) = -A_+ x(s)$. Similarly for $E^-$.

So you avoid the asymptotic analysis of 4.16! (but you still need 4.17).[14]

**Rmk.** *The same arguments show that the index of $\partial_s + A_s$ is invariant under homotopying the*[15] *$W^{1,2}$-path $u$ with fixed ends $p, q$. Just compare the two operators by parallel transport for small hpies, and repeat the above arguments.*

[14]Note: $\phi^s_{s_0} : E^+(s_0) \to E^+(s)$ is an isomorphism. So all we care about is $E^+(s)$ for large $s$.

[15]recall we trivialized our section $F$ on $u^*TM$ to obtain $\partial_s + A_s$.