LECTURE 17.

PART III, MORSE HOMOLOGY, 2011

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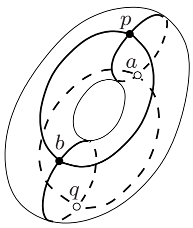
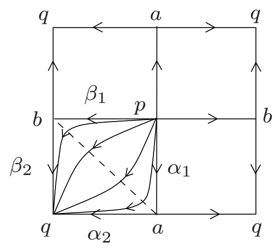
5. Compactness Theorem

5.1. **Motivating example.**
Consider $M = T^2$ tilted $\subset \mathbb{R}^3$ (see Hwk 1), $f =$ height function.

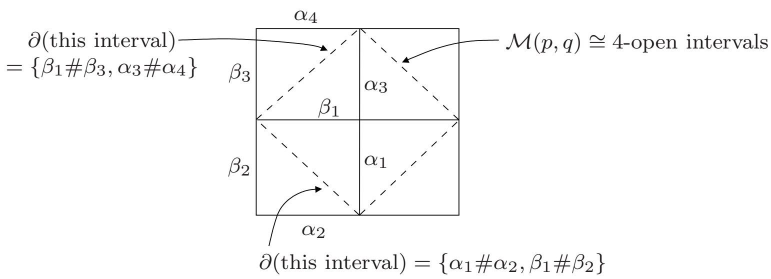




where we view the torus as a square with parallel sides identified. Observe:

$$
\begin{array}{ll}
\dim W(p,q) = |p| - |q| = 2 & \text{(parametrized flowlines)} \\
\dim \mathcal{M}(p,q) = 1 & \text{(unparametrized flowlines = trajectories)}
\end{array}
$$

Consider the lower left subsquare: the transverse intersection of the flowlines with the dashed subdiagonal parametrizes the space of trajectories $\mathcal{M}(p,q)$ (since you fix the parametrization by declaring that $u(0)$ is the point of intersection). Considering all squares, the four open subdiagonals parametrize $\mathcal{M}(p,q)$:



The boundaries of the open intervals involve broken trajectories, such as $\alpha_1\#\alpha_2$ which means the trajectory α_1 followed by α_2. Thus:

$$
\begin{array}{rcll}
\partial\mathcal{M}(p,q) & \cong & \mathcal{M}(p,a) \times \mathcal{M}(a,q) \sqcup \mathcal{M}(p,b) \times \mathcal{M}(b,q) & \text{(0-dimensional)} \\
& = & \text{8 broken trajectories} &
\end{array}
$$

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and the obvious way to compactify this is to compactify each interval separately:[1]

$$\begin{aligned}\overline{\mathcal{M}}(p,q) &= \mathcal{M}(p,q)\cup\partial\mathcal{M}(p,q) = \text{disjoint union of 4 closed intervals}\\ &= \text{compact smooth 1-manifold.}\end{aligned}$$

Key idea: to compactify $\mathbb{R}$ ($\cong$ open interval)

(1) identify which sequences fail to have a convergent subsequence:
$x_n \subset \mathbb{R}$ with $|x_n|\to\infty$

(2) artificially add the limit points to your set:
$\overline{\mathbb{R}} = \mathbb{R}\cup\{-\infty,+\infty\}$

(3) declare new open sets[2] to ensure those new points are indeed limit points:
$[-\infty,r),(r,+\infty]$ for all $r\in\mathbb{R}$

Upshot: $\overline{\mathbb{R}}$ is compact, whose *open part*[3] $\mathbb{R}\subset\overline{\mathbb{R}}$ has the original topology, and it has *boundary* $\partial\overline{\mathbb{R}} = \{\text{new points}\}$.

5.2. **The topology of $\mathcal{M}(p,q) = W(p,q)/\mathbb{R}$.** Recall that

$$W(p,q)\subset U\subset W^{1,2}_{loc}(\mathbb{R},M)\subset C^0_{loc}(\mathbb{R},M)$$

and $W(p,q)$ is a submanifold of U. So it comes with a topology induced from U which in particular implies C^0-convergence on compact sets.

Call $\pi : W(p,q)\to\mathcal{M}(p,q)/\mathbb{R}, u\mapsto\pi(u)=[u]$ the quotient map. The $\mathbb{R}$-action is just the reparametrization action given by shifting the s-variable by a constant:

$$v=[u]=[u(\cdot+\text{constant})].$$

We call such a $u=\widetilde{v}\in W(p,q)$ a ***lift*** of $v\in\mathcal{M}(p,q)$. The quotient topology is: $\mathcal{O}\subset\mathcal{M}(p,q)$ is open $\Leftrightarrow \pi^{-1}\mathcal{O}\subset W(p,q)$ is open. More usefully:

$$v_n\to v\in\mathcal{M}(p,q)\Leftrightarrow\exists\widetilde{v}_n\to\widetilde{v}\text{ in }W(p,q)$$

or even more explicitly:

$$\boxed{[u_n]\to[u]\text{ in }\mathcal{M}(p,q)\Leftrightarrow u_n(\cdot+s_n)\to u(\cdot)\text{ in }W(p,q)\text{ for some }s_n\in\mathbb{R}}$$

Example. $u\in W(p,q)\Rightarrow u_n(s)=u(s+n)$ not cgt in $W(p,q)$, but cgt to $[u]$ (indeed constant!) in $\mathcal{M}(p,q)$.

5.3. **C^0_{loc}-convergence.** M^m closed mfd, $f:M\to\mathbb{R}$ Morse.

Lemma. *$\partial_s u_n = -\nabla f(u_n)\Rightarrow\exists$ subseq $u_n\to u$ in C^0_{loc} ($=C^0$-cgce on compacts).*

Proof. $u_n|_{[-S,S]}:[-S,S]\to M\subset\mathbb{R}^a$ are equibounded (since M is compact) and equicontinuous by the mean value theorem using $|\partial_s u_n|\le\sup_{x\in M}|\nabla f(x)|$. Now apply Arzela-Ascoli (see 2.1). □

Cor. $\boxed{W=\bigcup_{p\neq q\in Crit(f)}W(p,q)\text{ is }C^0_{loc}\text{-compact.}}$

[1] You do not want to identify the ends of the intervals pairwise, giving a circle: consider two sequences of trajectories, one which converges to $\beta_1\#\beta_2$ and $\beta_1\#\beta_3$ respectively (for example). Then the two sequences are not actually approaching each other, because β_2 and β_3 are not equal. So you do not want to identify the two limiting broken trajectories.

[2] the new topology is then the one generated by the old topology and these new open sets. Example: $\{-\infty\}$ is a closed set because, for example, $\{-\infty\}=\overline{\mathbb{R}}\setminus((-\infty,0)\cup(-1,+\infty])$.

[3] $\overline{\mathbb{R}}$ is open & closed, $\mathbb{R}\subset\overline{\mathbb{R}}$ is open but no longer closed (since the complement of $\emptyset$ is now $\overline{\mathbb{R}}$).

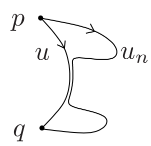
Rmk. *Suppose* $\left\{ \begin{smallmatrix} \partial_s u_n = -\nabla f(u_n) \\ \partial_s u = -\nabla f(u) \end{smallmatrix} \right\}$. *Then* $u_n \to u$ *in* $C^0_{loc} \Leftrightarrow$ *in* $C^k_{loc}\ \forall k \Leftrightarrow$ *in* C^∞_{loc}.

Proof. Usual bootstrapping: $u_n \to u$ in $C^0_{loc} \Rightarrow -\nabla f(u_n) \to -\nabla f(u)$ in C^0_{loc} $\Rightarrow u_n \to u$ in $C^1_{loc} \Rightarrow$ etc.

Cor. W *is* C^∞_{loc}*-compact.*

Thm. $\boxed{C^0_{loc}\text{-convergence in } W(p,q) \Rightarrow \text{convergence in } W(p,q) \text{ (in 5.1 topology)}}$

Proof. Suppose $u_n \to u$ in C^0_{loc}, where $u_n, u \in W(p,q)$. On $[-S,S]$ we have C^1 convergence hence $W^{1,2}$ convergence. So we reduce to showing convergence at the ends. The key trick is to show that for energy reasons, the u_n, u must be uniformly[4] close to the critical points at the ends, so this does not happen:



Recall $E(u) = \int_{\mathbb{R}} |\partial_s u|^2\, ds$. Pick $S \gg 0$ such that

$$\int_{\mathbb{R}\setminus[-S,S]} |\partial_s u|^2 = \frac{1}{2}\delta \quad \text{(small)}.$$

By Rmk:

$$\begin{array}{ll} \Rightarrow & u_n|_{[-S,S]} \to u|_{[-S,S]} \text{ in } C^1 \\ \Rightarrow & E(u_n|_{[-S,S]}) \to E(u|_{[-S,S]}) \\ \overset{\text{by } 3.2}{\Rightarrow} & E(u_n|_{\mathbb{R}\setminus[-S,S]}) \le \delta \quad (\text{ for } n \gg 0,\ \text{say } n \ge N) \\ \overset{\text{by } 3.3}{\Rightarrow} & u_n(s) \in \text{small ball around } p \text{ for } s \le -S, n \ge N \\ & u_n(s) \in \text{small ball around } q \text{ for } s \ge +S, n \ge N \end{array}$$

Where 3.2 refers to the a priori estimate $f(p)-f(q) = E(u) = E(u_n)$ and 3.3. refers to the No escape Lemma: if you have little energy left and you have to converge to p, q then you must be close to p, q.

Thus we reduce to showing $W^{1,2}$ convergence near p, q. WLOG do the case q.

Hwk 17 Exponential cgce at the ends: *Locally near* $q = 0$, $\exists c, \delta > 0$, $\exists$ *ball* $B_r(0)$ *s.t. if* $u : [S, \infty) \to B_r(0)$ *solves* $\partial_s u = -\nabla f(u)$ *then*

$$|u(s)| \le c \cdot e^{-\delta s} \quad \text{for } s \ge S.$$

So in our case, $u_n(s) \to q$ exponentially fast as $s \to \infty$ at a rate independent of n.

$\Rightarrow \operatorname{dist}(u_n(s), u(s)) \le \operatorname{dist}(u_n(s), q) + \operatorname{dist}(q, u(s)) \to 0$ exp fast in s.

$\Rightarrow -\nabla f(u_n(s)), -\nabla f(u(s))$ exp close, indeed exp close to $-\nabla f(q) = 0$

$\Rightarrow u_n, u$ are $W^{1,2}$-close on $[S, \infty)$. □

Rmk. *In more complicated situations, when you don't have a result like 3.2 in the above argument, you would* assume *that* $E(u_n)$ *is bounded. So the Thm would hold for sequences having a bound on the energies.*

[4] independently of n.

5.4. **Convergence to broken trajectories.** Given $u_n \in W(p,q)$. By Cor 5.2, a subsequence $u_n \to u$ in W in C^0_{loc}. Two situations can arise:

Case 1: no breaking. if $u \in W(p,q)$ then $u_n \to u$ in $W(p,q)$ by Thm 5.2.

Case 2: breaking. $u \in W(p_u, q_u) \neq W(p,q)$. Observe:

$f(p) \geq f(p_u) \geq f(q_u) \geq f(q)$ since f decreases along flowlines (3.2)

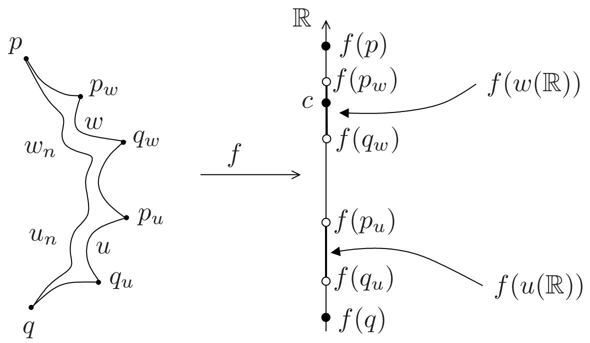
$\boxed{\text{Trick (3.3): } \exists \delta > 0 \text{ s.t. flowlines with ends near distinct crit pts consume } E \geq \delta}$

Lemma (Reparametrization Trick)**.** *Assuming we are in Case 2,*
$\exists s_n \in \mathbb{R}$ *such that* $w_n = u_n(\cdot + s_n) \overset{C^0_{loc}}{\to} w$ *with* $f(w(\mathbb{R})) \cap f(u(\mathbb{R})) = \emptyset$.

Proof. Suppose $p_u \neq p$ (case $q_u \neq q$ is similar). Then

$$f(p) > f(p_u),$$

otherwise $u_n|_{[-a_n,-b_n]}$ has ends close to p, p_u (for b_n large, a_n very large) with energy $\approx f(p) - f(p_u) = 0$ contradicting Trick 3.3.



Fix a regular value c of f with

$$f(p) > c > f(p_u)$$

and any $s_n \in \mathbb{R}$ with

$$f(u_n(s_n)) \to c.$$

Cor 5.2 $\Rightarrow w_n = u_n(\cdot + s_n) \overset{C^0_{loc}}{\to} w$ some $w \in W(p_w, q_w)$, with

$$f(p) \geq f(p_w) > c > f(q_w) \qquad (*).$$

We claim[5] that $s_n \to -\infty$. Proof:

$$\begin{array}{rcll} u_n \overset{C^0_{loc}}{\to} u & \Rightarrow & c > f(p_u) > f(u(-S)) \overset{n \gg 0}{\approx} f(u_n(-S)) & \\ & \Rightarrow & c > f(u_n|_{[-S,\infty)}) & \text{(for } n \gg 0) \\ & \Rightarrow & s_n \leq -S \ \checkmark & \text{(by } (*)) \end{array}$$

It remains to check $f(q_w) \geq f(p_u)$.

Suppose by contradiction $f(p_u) > f(q_w)$. Observe:

[5]intuition: it takes ∞ time to reach p_u since $\nabla f \to 0$.

$$\begin{array}{ccccc} & \overset{S \gg 0}{} & & \overset{n \gg 0}{} & \\ f(q_w) & \approx & f(w(S)) & \approx & f(u_n(S+s_n)) \\ \wedge & & & & \wedge \\ f(p_u) & \approx & f(u(-S)) & \approx & f(u_n(-S) \end{array}$$

hence this follows. But $s_n \to -\infty$ so contradicts f decreasing along u_n

$\square$