LECTURE 20.

PART III, MORSE HOMOLOGY, 2011

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6.3. **Invariance Theorem.** Let M be a closed mfd.

Thm. *$MH_*(M)$ does not depend on the auxiliary parameters you chose: given Morse $f_0, f_1 : M \to \mathbb{R}$ and generic metrics g_0, g_1, there is an isomorphism*

$$[\varphi_{10}] : MH_*(f_0, g_0) \xrightarrow{\cong} MH_*(f_1, g_1)$$

and these isomorphisms satisfy

(1) $[\varphi_{21}] \circ [\varphi_{10}] = [\varphi_{20}] \qquad (\forall f_i, g_i,\ i = 0, 1, 2)$
(2) $[\varphi_{00}] = \mathrm{id} \qquad (\forall f_0, g_0)$

Outline

(1) Construct a *continuation map*

$$\boxed{\varphi : MC_*^- \to MC_*^+} \; (MC_*^\pm = MC_*(f^\pm, g^\pm), f^\pm \text{ Morse}, g^\pm \text{ generic})$$

defined on generators as follows (then extend linearly):

$$\varphi(p^-) = \sum_{\dim \mathcal{C}(p^-,q^+)=0} \#\mathcal{C}(p^-, q^+) \cdot q^+$$

which counts the *moduli space of continuation solutions*

$$\boxed{\begin{aligned} \mathcal{C}(p^-, q^+) = \{v : \mathbb{R} \to M \quad : \quad & \partial_s v = -\nabla^s f_s(v), \\ & v(s) \to p^-, q^+ \text{ as } s \to -\infty, +\infty\} \end{aligned}}$$

where $p^- \in \mathrm{Crit}(f^-)$, $q^+ \in \mathrm{Crit}(f^+)$, $g_s(\nabla^s f_s, \cdot) = df_s$. This moduli space depends on a choice of smooth homotopy f_s, g_s,

$$s \mapsto (f_s : M \to \mathbb{R}), \quad s \mapsto g_s$$

where the functions f_s need not be Morse and the metrics g_s need not be Morse-Smale for f_s. The key requirement[1] is that for some S:

$$\boxed{f_s = \begin{cases} f^- & \text{for } s \le -S \\ f^+ & \text{for } s \ge S \end{cases} \qquad g_s = \begin{cases} g^- & \text{for } s \le -S \\ g^+ & \text{for } s \ge S \end{cases}} \quad (*)$$

Rmk. *Transversality for $\mathcal{C}(p^-, q^+)$ is achieved for generic paths g_s.*

Rmk. *Note that we do not[2] quotient $\mathcal{C}(p^-, q^+)$ by an $\mathbb{R}$-action by shifting s, unlike what we did for $\mathcal{M}(p, q) = W(p, q)/\mathbb{R}$.*

(2) $\varphi =$ identity for the constant data $f_s = f$, $g_s = g^+ = g^-$.

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[1] this is crucial for the energy estimate, later.

[2] indeed, cannot: $\nabla^s f_s$ is not invariant under shifting $s \mapsto s + \text{constant}$.

(3) φ is a chain map:

$$\boxed{\varphi \circ \partial^- = \partial^+ \circ \varphi} \qquad (\partial^\pm : MC_*^\pm \to MC_{*-1}^\pm)$$

hence we get a map on homology:

$$[\varphi] : MH_*^- \to MH_*^+.$$

(4) $\boxed{[\varphi] \text{ does not depend on } f_s, g_s}$

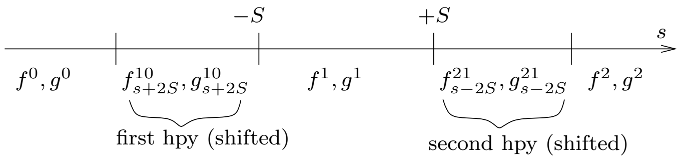
Consider a homotopy $(f_s^\lambda, g_s^\lambda)_{0\le\lambda\le 1}$ from f_s^0, g_s^0 to f_s^1, g_s^1, where we assume $(*)$ also for f_s^λ, g_s^λ. Denote φ^0, φ^1 the continuations for f_s^0, g_s^0 and f_s^1, g_s^1.

Claim φ^0, φ^1 are chain homotopic:

$$\boxed{\begin{array}{l} \exists\, K : MC_*^- \to MC_{*+1}^+ \\ \varphi^0 - \varphi^1 = K \circ \partial^- + \partial^+ \circ K \end{array}}$$

hence[3] $[\varphi^0] - [\varphi^1] = I$. ✓

(5) Suppose f_s^{10}, g_s^{10} is a homotopy from f^0, g^0 to f^1, g^1, and f_s^{21}, g_s^{21} is a homotopy from f^1, g^1 to f^2, g^2. Glue[4] the (reparametrized) homotopies:



Claim. For $S \gg 0$, the φ obtained for this glued homotopy equals the composite $\varphi^{21} \circ \varphi^{10}$ of the continuation maps for the two homotopies.

(6) **Consequences of these properties:**
(4) and (5) $\Rightarrow [\varphi^{21}] \circ [\varphi^{10}] = [\varphi^{20}]$ (independently of choices of hpies) ✓
(2) and (4) $\Rightarrow [\varphi^{00}] = \text{identity}$ (independently of choice of hpy) ✓
$\Rightarrow [\varphi^{01}] \circ [\varphi^{10}] = [\varphi^{00}] = \text{identity}$ so $[\varphi^{10}]$ injective
$\Rightarrow [\varphi^{10}] \circ [\varphi^{01}] = [\varphi^{11}] = \text{identity}$ so $[\varphi^{10}]$ surjective
$\Rightarrow [\varphi^{10}]$ isomorphism
$\Rightarrow$ Theorem

Key ideas in the proofs:

(1) Redo the transversality proof, now using:

$$G = \{C^k\text{-paths of metrics } s \mapsto g_s \text{ with } g_s = \begin{cases} g^- & \text{for } s \le -S \\ g^+ & \text{for } s \ge S \end{cases} \}$$
$$F(u, g_s) = \partial_s u - \nabla^s f_s(u) \quad (\text{where } g_s(\nabla^s f_s, \cdot) = df_s)$$

$\Rightarrow$ Parametric transversality, Fredholm analysis, etc. like we did for $W(\cdot,\cdot)$

$$\Rightarrow \boxed{\begin{array}{l} \mathcal{C}(p^-, q^+) \text{ smooth mfd for generic smooth } g_s \\ \dim \mathcal{C}(p^-, q^+) = |p| - |q| \end{array}}$$

In (3) we explain how to compactify $\mathcal{C}(p^-, q^+)$, and it shows that $\mathcal{C}(p^-, q^+)$ is compact when $\dim \mathcal{C}(p, q) = |p| - |q| = 0$. So φ is well-defined.

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[3] since $\partial^- = 0$ on $\ker \partial^-$, $\partial^+(K\bullet) = 0$ modulo $\operatorname{im} \partial^+$.

[4] for large S these glue correctly.

(2) For constant data $f = f_s$, $g = g_s$,

$$\mathcal{C}(p,q) = W(p,q).$$

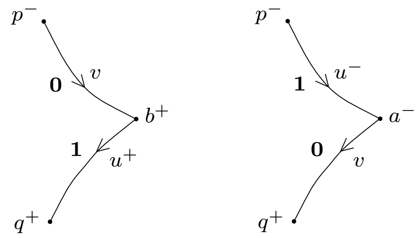
So since $g = g^- = g^+$ is generic, $W(p,q)$ is a smooth mfd, thus so is $\mathcal{C}(p,q)$. Finally we make a dimension argument:

If $v \in \mathcal{C}(p,q)$ is a solution then $v(\cdot + \text{constant})$ is a solution since $\partial_s v = -\nabla f(v)$ has ∇f independent of s.
$\Rightarrow$ if v non-constant, then there is a 1-family of solutions $v(\cdot + \text{constant})$
$\Rightarrow \dim \mathcal{C}(p,q) \geq 1$ if $p \neq q$
$\Rightarrow \varphi$ only counts constant solutions $\mathcal{C}(p,p) = W(p,p) = \{\text{constant at } p\}$
$\Rightarrow \varphi(p) = p$
$\Rightarrow \varphi = \text{identity}$.

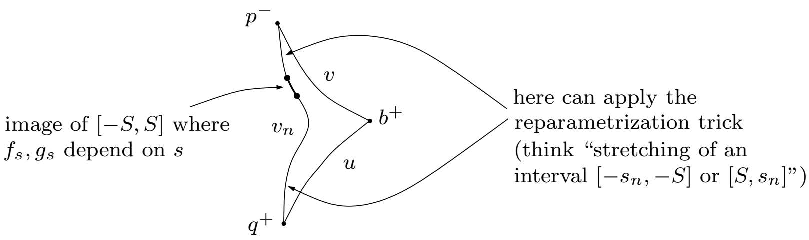
Rmk. *The key is* not *to make an s-dependent perturbation of $g_s = g^- = g^+$, but rather to perturb s-independently (in fact, since we assume $g^- = g^+$ is generic, we don't need to). This gives transversality for $W(p,q) = \mathcal{C}(p,q)$.*

(3) Study the breaking of 1-dimensional $\mathcal{C}(p^-, q^+)$, so $|p^-| - |q^+| = 1$.

The key claim is that a once-broken continuation solution does not consist of two continuation solutions, but rather consists of one continuation solution and one $f^\pm$-trajectory:[5]



Proof. Consider a sequence v_n of continuation solutions in $\mathcal{C}(p^-, q^+)$. Consider the interval $[-S, S]$ where f_s, g_s depend on s. Since $[-S, S]$ is compact, by Arzela-Ascoli a subsequence will satisfy C^0-convergence on $[-S, S]$ so there is no breaking there.



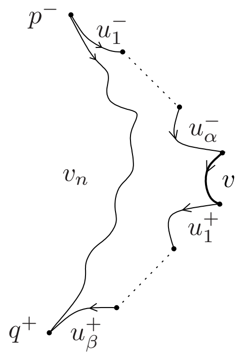
[5] In the figure, we denote by v the continuation solutions, and by $u^\pm$ the $-\nabla^\pm f^\pm$-trajectories. We write boldface numbers which indicate the Fredholm index of the linearization of the Fredholm section. So in the figure, $|b| - |q| = 1$, $|p| - |a| = 1$. Recall that

$$\dim(\mathcal{C} \text{ or } W \text{ spaces}) = \dim \text{ tangent space} = \dim \ker(\text{surj Fred operator}) = \text{index}.$$

For W spaces, dimension 1 implies that $\mathcal{M} = W/\mathbb{R}$ has dimension 0. So solutions v, u in $\mathcal{C}, W$ spaces of dim $0, 1$ respectively are called ***rigid*** (or *isolated*).

Finally, these dimension numbers add to give the correct dimension of the breaking family because of linear gluing (see 5.5 Step 2, details to ①): "gluing kernels of Fred operators is iso to kernel of glued Fred operator". So indices add correctly under gluing.

In the general case (when we do not assume $|p|-|q|=1$): mimick the proof of 5.3 and use the above observation $\Rightarrow$ general breaking for $\mathcal{C}(p^-,q^+)$ is:



Here u_i^- are $-\nabla^- f^-$-trajectories, u_j^+ are $-\nabla^+ f^+$-trajectories, and v is a continuation map for (f_s, g_s).

Details. Reviewing the proof of compactness for W spaces, observe that what we needed crucially was an a priori energy estimate. In our case it is:

$$\begin{array}{rcl} E(v) & = & \int_{-\infty}^{\infty} |\partial_s v|^2 \, ds \\ & = & \int g_s(\partial_s v, \partial_s v)\, ds \\ & = & -\int df_s(\partial_s v)\, ds \quad (\text{since } \partial_s v = -\nabla^s f_s) \\ & = & -\int (\partial_s(f_s \circ v) - (\partial_s f_s)(v))\, ds \\ & \leq & f^-(p^-) - f^+(q^+) + \int |\partial_s f_s|_v \, ds \\ & \leq & f^-(p^-) - f^+(q^+) + 2S \cdot \max\limits_{x\in M} |\partial_s f_s(x)| \end{array}$$

We also needed the energy consumption trick 3.3. This can also be used in our setup in the regions $s \leq -S$, $s \geq S$ where f_s, g_s do not depend on s.

Key observation: each u_i^+, u_j^- contributes to 1 to the index difference $|p^-| - |q^+|$, since the $\mathcal{M}^\pm$ spaces are empty if the index difference of the ends is zero or negative.

Key $\Rightarrow$ for $|p^-| - |q^+| = 0$ no breaking can occur $\Rightarrow \mathcal{C}(p^-,q^+)$ is compact.
Key $\Rightarrow$ for $|p^-| - |q^+| = 1$ only 1 breaking can occur for dimension reasons.

Hence (after reproving the gluing theorem) for $|p| - |q| = 1$:

$$\partial\overline{\mathcal{C}}(p^-,q^+) = \bigsqcup_{a^-} \mathcal{M}_0^-(p,a) \times \mathcal{C}_0(a,q) \cup \bigsqcup_{b^+} \mathcal{C}_0(p,b) \times \mathcal{M}_0^+(b,q)$$

where the numbers indicate the dimension we request[6] and $\mathcal{M}^\pm$ are the $\mathcal{M}$ spaces for $(f^\pm, g^\pm)$.

$$\begin{array}{l} \Rightarrow \overline{\mathcal{C}}(p,q) \text{ compact 1-mfd} \\ \Rightarrow \#\partial\overline{\mathcal{C}}(p,q) \text{ is even} \\ \Rightarrow \varphi \circ \partial^- + \partial^+ \circ \varphi = 0. \quad \square \end{array}$$

(4) $(f_s^\lambda, g_s^\lambda)_{0\leq\lambda\leq 1}$ is called *homotopy of homotopies* $(*)$

[6] $|p| = |b| = k$, $|a| = |q| = k-1$.

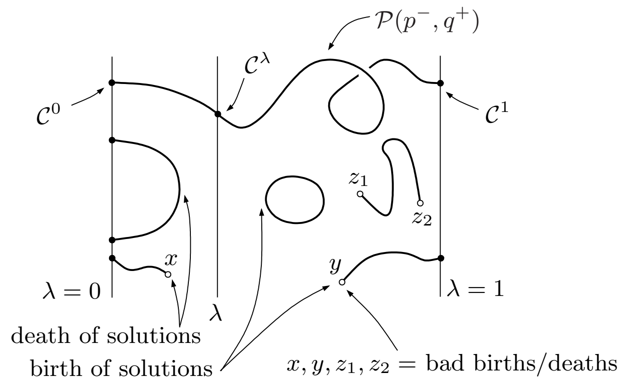
Fix p^-, q^+ with $|p^-| - |q^+| = 0$. Look at the "movie"

$$\boxed{\mathcal{C}^\lambda = \mathcal{C}^\lambda(p^-, q^+; f_s^\lambda, g_s^\lambda)}$$

as λ varies. This "movie" is called the *parametrized moduli space*

$$\mathcal{P}(p^-, q^+) = \bigsqcup_{0 \le \lambda \le 1} \mathcal{C}^\lambda$$

For generic data $(*)$, it is a smooth 1-mfd:[7]



Warning. *$\mathcal{C}^\lambda$ may not be a smooth manifold for fixed λ. Genericity of the family (in λ) does not guarantee genericity of each point of the family (fixed $\lambda = \lambda_0$). However, one can guarantee that each $\mathcal{C}^\lambda$ satisfies transversality except for finitely many values of λ.*

Breaking analysis: a subsequence (λ_n, v_n) has $\lambda_n \to 0, 1$ or $\lambda_0 \in (0, 1)$.

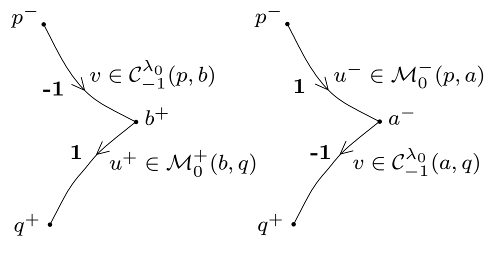
$$\Rightarrow \partial\overline{\mathcal{P}} = \mathcal{C}^0 \sqcup \mathcal{C}^1 \sqcup B$$

where $B = \{\text{bad births/deaths}\}$.

If $B = \emptyset$, then P is a 1-cobordism from $\mathcal{C}^0$ to $\mathcal{C}^1$,

$$\Rightarrow \#\mathcal{C}^0 - \#\mathcal{C}^1 = \#\partial P = \text{even} = 0 \bmod 2$$
$$\Rightarrow \varphi^0 = \varphi^1$$

If $B \neq \emptyset$, let K count the bad set B:



Question. How is it possible that such v exist: the relevant moduli space $\mathcal{C}^{\lambda_0}$ is negative dimensional!

[7] $\dim \mathcal{P}(p, q) = |p| - |q| + 1$, where the additional 1 is because of the parameter λ.

Answer. This happens because $f_s^{\lambda_0}, g_s^{\lambda_0}$ is not generic.[8] So "-1" is the *virtual dimension*: the dimension you would get *if* transversality held true:

$$\text{virdim}\,\mathcal{C}^{\lambda_0}(p,q) = |p| - |q|.$$

Def. *Such* $v \in \mathcal{C}^{\lambda_0}_{-1}(\cdot,\cdot)$ *(virtual dimension* -1*) are called* rogue trajectories.

There are no rogue trajectories at $\lambda = 0, 1$ since by assumption f_s^0, g_s^0 and f_s^1, g_s^1 are generic. So define

$$\boxed{\begin{aligned} K \;:\; & MC_*^- \to MC_{*+1}^+ \\ & Kx^- = \sum_{|y^+|=|x^-|+1} \#(\text{rogue trajectories from } x \text{ to } y)\cdot y^+ \end{aligned}}$$

So in the above pictures, the contributions would be:

$$\begin{aligned} Kp^- &= b^+ + \cdots \\ Ka^- &= q^+ + \cdots \\ \partial^- p^- &= a^- + \cdots \\ \partial^+ b^+ &= q^+ + \cdots \end{aligned}$$

So $\boxed{\varphi^0 - \varphi^1 = \partial^+ \circ K + K \circ \partial^-}$ comes from counting the even number of elements in:

$$\begin{aligned} \partial\overline{\mathcal{P}}(p^-, q^+) = \mathcal{C}^0 \sqcup \mathcal{C}^1 \quad & \sqcup \bigsqcup_{\lambda_0\in(0,1),\, b^+\in \text{Crit}\, f^+} \mathcal{C}^{\lambda_0}_{-1}(p,b) \times \mathcal{M}_0^+(b,q) \\ & \sqcup \bigsqcup_{\lambda_0\in(0,1),\, a^-\in \text{Crit}\, f^-} \mathcal{M}_0^-(p,a) \times \mathcal{C}^{\lambda_0}_{-1}(a,q) \end{aligned}$$

(5) This is a gluing argument: you can *approximately* glue solutions, then for large S (depending on p, r, q) you can associate a "unique" actual solution. This produces a bijection:

$$\bigsqcup_{q^1\in \text{Crit} f^1} \mathcal{C}_0(p^0, q^1; 1^{\text{st}} \text{ hpy}) \times \mathcal{C}_0(q^1, r^2; 2^{\text{nd}} \text{ hpy}) \to \mathcal{C}_0(p^0, r^2; \text{glued hpy})$$

So $\varphi^{21} \circ \varphi^{10}(p^0)$ and $\varphi^{20}(p^0)$ have the same r^2 coefficients. Therefore

$$\varphi^{21} \circ \varphi^{10} = \varphi^{20}$$

(there are only finitely many critical points, so you can pick the largest of the S's, as you vary p, q, r).[9]

[8] Just because the family $(*)$ is generic, does not mean that each $f_s^{\lambda_0}, g_s^{\lambda_0}$ is generic.

[9] *Non-examinable:* In more complicated situations, when there are infinitely many generators, you can still prove the equation at the level of homology: cycles involve *finite* linear combinations of generators, so only finitely many generators are involved in showing that the two expressions agree on a given cycle.