LECTURE 21.

PART III, MORSE HOMOLOGY, 2011

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7. Applications

7.1. **Poincaré duality.** Define

$$MC^* = \text{ Morse cochain complex } = \sum_{p \in \mathrm{Crit}\, f} \mathbb{Z}/2 \cdot p$$

$$\delta p = \sum_{\dim \mathcal{M}(q,p)=0,\ q \neq p} \#\mathcal{M}(q,p) \cdot q \quad \text{(note the changed order!)}$$

Cor. *$MH^*(f) = H_*(MC^*, \delta) =$ Morse cohomology $\cong H^*(M)$ (proved like 6.1 (3)).*

Thm. *$MH_*(f) \cong MH^{m-*}(-f)$*

Proof. Note that $\mathrm{ind}_{-f}(p) = m - \mathrm{ind}_f(p)$. So define on generators

$$\phi : MC_*(f) \to MC^{m-*}(-f),\ p \mapsto p$$

and extend linearly. So ϕ is clearly an isomorphism of vector spaces. Moreover,

$$\begin{array}{ccc} \mathcal{M}(p,q;f) & \cong & \mathcal{M}(q,p;-f) \\ [u(s)] & \mapsto & [u(-s)] \end{array}$$

so ϕ is a chain map:

$$(\phi \circ \partial)(p) = \sum \#\mathcal{M}(p,q;f) \cdot q = \sum \#\mathcal{M}(q,p;-f) \cdot q = (\delta \circ \phi)(p). \quad \square$$

7.2. **Künneth's theorem.**

Thm. *Over $\mathbb{Z}/2$ coefficients,*

$$\begin{tikzcd} MH_*(f_1 \oplus f_2) \arrow[r, "\cong"] & MH_*(f_1) \otimes MH_*(f_2) \arrow[d, "\cong"] \\ H_*(M_1 \times M_2) \arrow[u, "\cong"] & H_*(M_1) \otimes H_*(M_2) \end{tikzcd}$$

where $f_1 : M_1 \to \mathbb{R}$, $f_2 : M_2 \to \mathbb{R}$ are Morse functions on closed mfds.

Proof.

$$\begin{array}{cccc} f_1 \oplus f_2 : & M_1 \times M_2 & \to & \mathbb{R} \\ & (m_1, m_2) & \mapsto & f_1(m_1) + f_2(m_2) \end{array}$$

So $d(f_1 \oplus f_2) = df_1 \oplus df_2 : TM_1 \oplus TM_2 \to \mathbb{R}$.

Also f_1, f_2 Morse $\Rightarrow$ $f_1 \oplus f_2$ Morse.

For g_1, g_2 generic metrics, we will use[1] $g_1 \oplus g_2$ on $M_1 \times M_2$. Then

$$W(p_1 \times p_2, q_1 \times q_2, f_1 \oplus f_2) = W(p_1, q_1, f_1) \times W(p_2, q_2, f_2)$$

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[1]We will see directly that transversality holds for $(f_1 \oplus f_2, g_1 \oplus g_2)$.

since $\partial_s u_1 \oplus \partial_s u_2 = -\nabla f_1 \oplus -\nabla f_2$. The grading is:

$$|p_1 \times p_2| = |p_1| + |p_2|.$$

To get 1-dimensional moduli spaces $W(\cdot,\cdot)$: need u_1 or u_2 constant, otherwise have $\dim \geq 2$ by shifting s in u_1 and in u_2.

$\Rightarrow d$ counts $\begin{cases} \mathcal{M}(p_1 \times p_2, q_1 \times p_2) = \mathcal{M}(p_1, q_1, f_1) \times \{p_2\} \\ \mathcal{M}(p_1 \times p_2, p_1 \times q_2) = \{p_1\} \times \mathcal{M}(p_2, q_2, f_2) \end{cases}$

$\Rightarrow d(p_1 \times p_2) = \partial p_1 \times p_2 + p_1 \times \partial p_2$

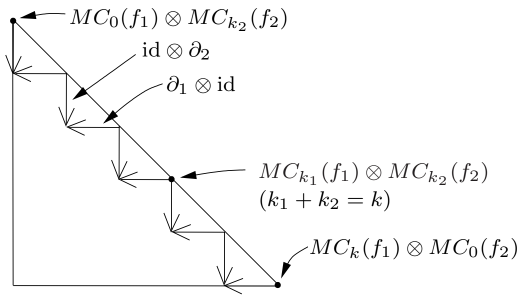
$$\Rightarrow MC_k(f_1 \oplus f_2) \cong \bigoplus_{k_1+k_2=k} MC_{k_1}(f_1) \oplus MC_{k_2}(f_2)$$

is an isomorphism of chain complexes, where the differential on the right is

$$d = \partial_1 \times \mathrm{id} + \mathrm{id} \times \partial_2$$

and this complex is denoted $(MC_*(f_1) \otimes MC_*(f_2))_{\text{grading } k}$.

Hence, by abstract algebra,[2] $MH_*(f_1 \oplus f_2) \cong MH_*(f_1) \otimes MH_*(f_2)$. □



7.3. **Morse-inequalities.** Abbreviate:

$$\begin{aligned} c_k &= \#(\text{crit pts of Morse } f \text{ of index } k) = \dim MC_k(f) \\ m_k &= \dim MH_k(f) \\ b_k &= \dim H_k(M) \text{ (Betti numbers)} \end{aligned}$$

computed using $\mathbb{Z}/2$ coefficients (although everything in this Section holds also if you use $\mathbb{Z}$ or $\mathbb{R}$ coefficients[3]).

Cor. $c_k \geq b_k$

Proof. $c_k = \dim MC_k \geq m_k = b_k$ (since $MH_k \cong H_k(M)$). □

Lemma. *If* $0 \longrightarrow C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C_n \longrightarrow 0$, $d_{i+1} \circ d_i = 0$, *where* C_i *are vector spaces over some field, then the Euler-characteristics agree:*

$$\chi(H_*) = \sum (-1)^i \dim H_i = \sum (-1)^i \dim C_i = \chi(C_*)$$

where H_* *is the homology of* (C_*, d).

[2] see May, *Concise Course in Algebraic Topology*, p.131: this is an abstract version of Künneth's theorem, and it uses the fact that we work over $\mathbb{Z}/2$.

[3] this involves orienting the moduli spaces $\mathcal{M}(p,q)$, and in the definition of ∂ you count the elements of $\mathcal{M}(p,q)$ with orientation signs ± 1. When using $\mathbb{Z}$, replace dim by rank.

Proof. $K_i = \ker d_i$, $J_{i+1} = \operatorname{im} d_i \subset C_{i+1}$,

$$C_i \cong K_i \oplus \text{complement} \xrightarrow{d_i} J_{i+1} \subset C_{i+1}$$

and the complement maps isomorphically onto J_{i+1}. So, using $H_i = K_i/J_i$

$$\begin{array}{rcl} \dim C_i &=& \dim K_i + \dim J_{i+1} \\ \dim H_i &=& \dim K_i - \dim J_i \end{array}$$

Writing c_i, h_i, k_i, j_i for the dimensions,

$$\sum_i (-1)^i c_i = \sum_i (-1)^i k_i + \sum_i (-1)^i j_{i+1} = \sum_i (-1)^i k_i - \sum_\ell (-1)^\ell j_\ell = \sum_\ell (-1)^\ell h_\ell$$

□

Thm (Morse inequalities)**.**

$$c_k - c_{k-1} + \cdots + (-1)^k c_0 \geq b_k - b_{k-1} + \cdots + (-1)^k b_0 \qquad \forall k$$

and for $k = m$ get equality: $\sum (-1)^i c_i = \sum (-1)^i m_i = \chi(M)$.

Proof.

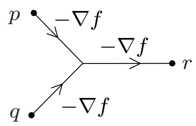
$$0 \longrightarrow MC_0 \xrightarrow{\partial_0} MC_1 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{k-1}} MC_k \xrightarrow{\partial_k} \operatorname{im}(\partial_k) \longrightarrow 0$$

So by the Lemma,

$$c_0 - c_1 + \cdots + (-1)^k c_k + (-1)^{k+1} \dim(\operatorname{im} \partial_k) = m_0 - m_1 + \cdots + (-1)^k m_k.$$

Now multiply by $(-1)^k$ and use $(-1)^{2k+1} \dim(\operatorname{im} \partial_k) \leq 0$. Finally use $m_i = b_i$. □

7.4. **Products.** You would like to count the following rigid configurations:



which is an abbreviation for:

$$\begin{array}{l} u_1 : (-\infty, 0] \to M,\ \partial_s u_1 = -\nabla f \\ u_2 : (-\infty, 0] \to M,\ \partial_s u_2 = -\nabla f \\ u_3 : [0, +\infty) \to M,\ \partial_s u_3 = -\nabla f \\ u_1(0) = u_2(0) = u_3(0). \end{array}$$

Note the intersection point lies in

$$W(p, q; r) = W^u(p) \cap W^u(q) \cap W^s(r). \qquad (*)$$

Problems: *Firstly, such configurations are unlikely: flowlines do not intersect unless they coincide. Secondly, we need $W(p, q; r)$ to be a manifold. To fix both these issues, one needs to perturb the function f on each edge,*[4] *so we use Morse f_1, f_2, f_3 on the 3 edges, and we pick a Morse-Smale metric g for all three edges.*[5]

So, for generic f_i, $W(p, q; r)$ is a manifold of codimension

$$\begin{array}{rcl} \operatorname{codim} W(p, q; r) &=& \sum \text{codims} \\ &=& m - |p| + m - |q| + |r| \end{array}$$

[4] in fact, it is enough to perturb f on only one of the first two edges, since we already have $W^u(p) \pitchfork W^s(r)$. But for more general graphs, you will have to perturb all f's.

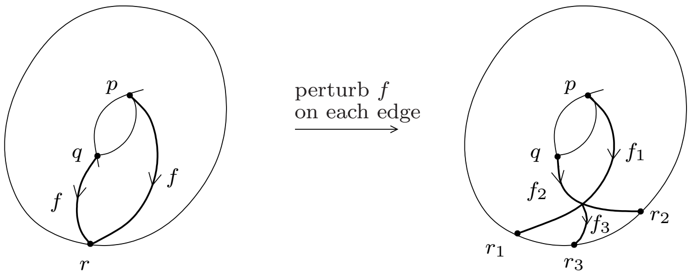
[5] a finite intersection of generic sets is generic.

we want this to be m (so we get $\dim = 0$) so want $|r| = |p| + |q| - m$. Thus

$$MC_a \otimes MC_b \xrightarrow{\psi} MC_{a+b-m}$$
$$p \cdot q = \psi(p,q) = \sum_{\dim W(p,q;r)=0} \#W(p,q;r) \cdot r$$

on generators, and extend bilinearly.

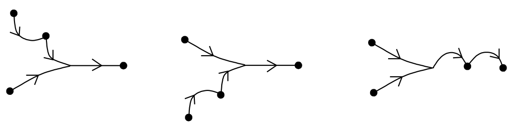
Example. *For the torus, you can check directly that $p \cdot q = r$ in the figure:*



which is Poincaré dual to the statement $H^1_{dR} \otimes H^1_{dR} \to H^2_{dR}$, $ds \wedge dt =$ (area form).

Claim. *ψ is a chain map*

Proof. Breaking/gluing analysis:



$\Rightarrow \psi \circ (\partial_1 \otimes \mathrm{id} + \mathrm{id} \otimes \partial_2) = \partial_3 \circ \psi$ ☐

Cor.

$$\begin{tikzcd} MH_a \otimes MH_b \arrow[r, "\cdot", "\textit{intersection product}"'] \arrow[d, "P.D."', "\cong"] & MH_{a+b-m} \arrow[d, "P.D."', "\cong"] \\ MH^\alpha \otimes MH^\beta \arrow[r, "\cup", "\textit{cup product}"'] & MH^{\alpha+\beta} \end{tikzcd}$$

where $\alpha = m - a$, $\beta = m - b$, $\alpha + \beta = m - (a + b - m)$.

Comparison with algebraic topology:

Claim. ψ is compatible with homotoping f_1, f_2, f_3.

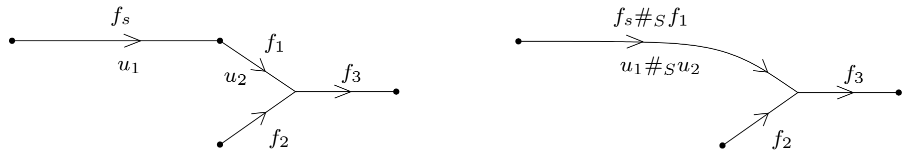
Consequence. WLOG f_1, f_2, f_3 are self-indexing. Then $MC_* \to C_*^{\mathrm{cell}}$, $p \mapsto W^u(p)$ is an isomorphism (see 3.10), and the cellular intersection product is:

$$W^u(p) \cdot W^u(q) = \sum_r \#(W^u(p) \pitchfork \widetilde{W^u(q)} \pitchfork \widetilde{W^s(r)}) \cdot W^u(r)$$

where you homotope $W^u(q), W^s(r)$ to $\widetilde{W^u(q)}$, $\widetilde{W^s(r)}$ to ensure the intersection is transverse and that all intersections occur away from the boundaries[6] of the unstable/stable mfds. So in fact, we do not need to homotope if we choose f_1, f_2, f_3 generically. This proves you recover the classical intersection product. ☐

[6]more precisely, we mean: away from the boundary of the compactification of the image. *Non-examinable:* the notion of *pseudo-manifolds* makes these arguments very rigorous.

Sketch of Proof of the Claim. Consider $\psi \circ (\varphi \circ \mathrm{id})$, where φ is a continuation map (so we are only homotopying f_1, using a hpy f_s from f_0 to f_1). Consider the following two configurations:



On the left, are the broken solutions counted by $\psi_{f_1,f_2,f_3} \circ (\varphi \circ \mathrm{id})$ where we emphasize for which f's ψ is defined. On the right, are the glued solutions counted by $\psi_{f_s \#_S f_1, f_2, f_3}$. By a gluing argument, for a large gluing parameter S, there is a bijection between the broken solutions and the glued solutions.

Now homotopying $f_s \#_S f_1$ to f_0, produces a chain homotopy K like in 6.3 (4):

$$\psi_{f_s \#_S f_1, f_2, f_3} - \psi_{f_0, f_2, f_3} = \partial_3 \circ K + K \circ (\partial_0 \otimes \mathrm{id} + \mathrm{id} \otimes \partial_2).$$

So the maps agree on homology. So $\psi_{f_1,f_2,f_3} \circ (\varphi \circ \mathrm{id}) = \psi_{f_0,f_2,f_3}$ on homology. $\square$