LECTURE 22 AND 23.

PART III, MORSE HOMOLOGY, 2011

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7.5. Spectral sequences.

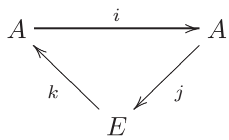
Spectral sequence = algebraic gadget that interlocks a bunch of exact sequences. It usually arises for a chain complex C_* with

$$d = d_0 + d_1 + d_2 + \cdots$$

where d_0 is "dominant" in some way over the higher order terms $d_1, d_2, \ldots$, and we hope to approximate $H_*(C_*, d)$ by

$$E^1 = H_*(C_*, d_0),\ E^2 = \text{“}H_*(E^1, d_1)\text{”},\ \ldots \overset{\text{cges?}}{\Rightarrow} H_*(C_*, d).$$

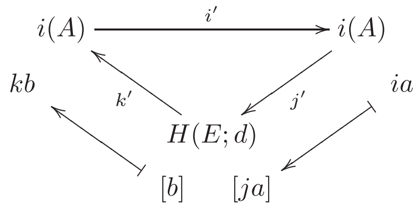
Def. *An* exact couple *is an exact*[1] *triangle of vector spaces of the form*



Given an exact couple, define

$$\boxed{d = j \circ k : E \to E}$$

Then $d^2 = jkjk = 0$ since $kj = 0$ by exactness. Thus we obtain the *derived couple*:



Exercise. Check these maps are well-defined, and that this new triangle is exact.

Rmk. *If i = inclusion, then $k = 0$, so $d = 0$, so $E \equiv H(E, d) = A/iA$ unchanged!*

7.6. Example: the spectral sequence for a bounded filtration.

Suppose (C_*, d) is a $\mathbb{Z}$-graded chain complex[2] with a filtration by subcomplexes[3]

$$0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_n = C_*$$

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[1]recall *exact* means the kernel of one arrow equals the image of the previous arrow.

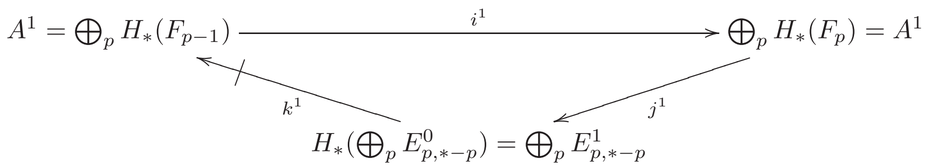
[2]$d : C_k \to C_{k-1}$, $d^2 = 0$.

[3]$dF_p \subset F_p$.

$\Rightarrow 0 \to F_{p-1} \xrightarrow{i} F_p \xrightarrow{j} F_p/F_{p-1} \to 0$ exact
$\Rightarrow$ define $F_p = 0$ for $p < 0$, $F_p = C_*$ for $p \geq n$. Then define

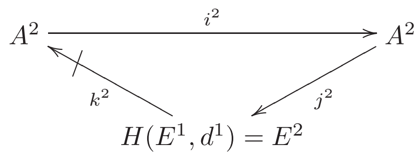
$$E^0_{p,*-p} = \bigoplus_p (F_p/F_{p-1})_{\text{the part in } \mathbb{Z}\text{-grading } *}$$

Then the LES asssociated to the above SES:[4]

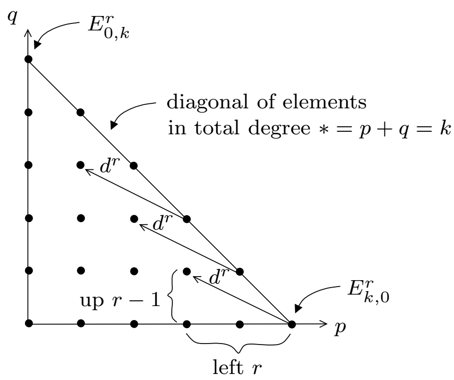


The dash on the arrow indicates that the grading $*$ drops by 1. Abbreviate $q = * - p$ ($* = p + q$ is called the *total degree*). Deriving the couple, we obtain:

$$d^1 = k^1 \circ j^1 : E^1_{p,q} \to E^1_{p-1,(*-1)-(p-1)} = E^1_{p-1,q}.$$



and keep deriving. So obtain $\boxed{E^r,\ d^r = j^r \circ k^r : E^r_{p,q} \to E^r_{p-r,q+r-1}}$



$$A^1 = \text{sum up } (0 \to H(F_0) \xrightarrow{i} H(F_1) \xrightarrow{i} \cdots \xrightarrow{i} H(F_n) \xrightarrow{=} H \xrightarrow{=} \cdots)$$

where $H = H(C_*, d) = H(F_p)$ for $p \geq n$. The image under i^n becomes:

$$\begin{aligned} A^{n+1} &= \text{sum up } (0 \to i^n H(F_0) \subset i^n H(F_1) \subset \cdots \subset i^n H(F_n) = H = \cdots) \\ &= \text{sum up } (G_0 \subset G_1 \subset \cdots \subset G_n = H = \cdots) \ \text{ where } G_p = \operatorname{im}(H(F_p) \xrightarrow{i^n} H). \end{aligned}$$

$\overset{\text{Rmk}}{\Rightarrow} E^r_{p,*-p} = \oplus (G_p/G_{p-1})_{\text{the part in } \mathbb{Z}\text{-grading } *} = E^\infty_{p,*-p}$ constant for $r \gg 0$.

But now $H \cong G_0 \oplus (G_1/G_0) \oplus (G_2/G_1) \oplus \cdots \oplus (G_n/G_{n-1})$, so rewriting:

$$\Rightarrow \boxed{H_*(C_*, d) \cong \bigoplus_p E^\infty_{p,*-p}}$$

[4]recall that every short exact sequence gives rise to a long exact sequence. E.g. see Hatcher's *Algebraic Topology*.

One abbreviates this result by writing

$$\boxed{E^1_{p,q} \Rightarrow H_*(C_*, d)}$$

(read "⇒" as "converges to") and one says the *spectral sequence* $E^1_{p,q}$ *converges.*[5]

Warning: the last two isomorphisms are not canonical, because you are recovering the group from certain successive quotients.

7.7. **Leray-Serre spectral sequence.**

Thm.

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \pi \\ & & B \end{array}$$

Let E be a fibre bundle with simply connected base B, and with fibre F, where B, E, F are closed mfds. Then there is a spectral sequence

$$E^2_{p,q} = MH_p(B) \otimes MH_q(F) \Rightarrow MH_*(E).$$

Example. *Künneth's theorem: $E = B \times F$, then $E^2_{p,q} = E^\infty_{p,q}$.*

Proof. Fix Morse-Smale data:

$$\begin{array}{ll} (b : B \to \mathbb{R}, g_B) & (f : F \to \mathbb{R}, g_F) \\ \operatorname{Crit} b = \{b_1, b_2, \ldots, b_n\} & \operatorname{Crit} f = \{y_1, \ldots, y_m\}. \end{array}$$

Pick disjoint opens B_i around $b_i \in B$ with trivializations

$$\begin{array}{ccc} E|_{B_i} & \xrightarrow{\cong} & B_i \times F \\ \downarrow f_i & & \downarrow f \\ \mathbb{R} & = & \mathbb{R} \end{array}$$

Fix bump functions $\rho_i : B \to [0,1]$, $\rho_i = \begin{cases} 0 & \text{outside } B_i \\ 1 & \text{near } b_i \end{cases}$

Then we obtain a function on E:

$$h = b + \varepsilon \sum \rho_i f_i : E \to \mathbb{R}$$

where we abusively write b but mean $b \circ \pi : E \to B \to \mathbb{R}$.

Claim. *h is Morse for $0 < \varepsilon \ll 1$.*

Proof. $h = b \oplus \varepsilon f$ on $(\rho_i = 1) \subset B_i \times F$ is Morse (compare Künneth proof) ✓

Outside $\cup_i(\rho_i = 1)$: $|db| > \delta > 0$, so for $\varepsilon \ll \delta$ get $|db| > \frac{1}{2}\delta > 0$ (ρ_i, f_i are C^1-bdd since F compact). □

This also proves that

$$\operatorname{Crit} h = \operatorname{Crit} b \times \operatorname{Crit} f \quad \text{(in the trivializations).} \qquad (*)$$

Now want to build a metric on E such that in the above trivializations we are in the Künneth setup:

$$\boxed{\begin{array}{rcl} g_E & = & g_B \oplus g_F \text{ on } B_i \times F \\ \Rightarrow \nabla h & = & \nabla b \oplus \varepsilon \nabla f_i \text{ on } \rho_i = 1. \end{array}}$$

Problem: *∇h is useless: outside $\rho_i = 1$ you get $\nabla \rho_i$ terms and also you will need to perturb g_E to get transversality*

[5]In our case, one also says the spectral sequence *degenerates* at sheet $n+1$ because $d^r = 0$ for $r \geq n+1$, so we may identify $E^{n+1} = E^{n+2} = \cdots$.

$\Rightarrow$ *you have no idea what $d\pi(\nabla h)$ is.*
$\Rightarrow$ *no idea what $\pi \circ (-\nabla h$ trajectory) is.*

Trick: we will construct a gradient-like vector field v for h such that

$$\boxed{\begin{cases} ① \ \ d\pi \circ v = \nabla b \\ ② \ \ v = \nabla h = \nabla b \oplus \varepsilon \nabla f_i \text{ on } \rho_i = 1 \end{cases}}$$

Hence, for $e \neq e' \in \operatorname{Crit} h$ define:

$$\boxed{V(e,e') = \{-v \text{ flowlines converging to } e, e'\}/\mathbb{R}}.$$

Because of ①, $V(e,e')$ projects via π to the moduli spaces $\mathcal{M}(b_i,b_j)$ for $b : B \to \mathbb{R}$, where $b_i = \pi(e)$, $b_j = \pi(e')$. Like for the $\mathcal{M}$ spaces,[6]

$$\boxed{\dim V(e,e') = |e| - |e'| - 1}$$

calculating the indices for h, since near the ends $-v = -\nabla h$ by ②.

Modifying g_E: Define the vertical and horizontal subspaces of TE by

$$\begin{aligned} V &= \ker d\pi, \\ H &= V^{\perp} \qquad \text{(perpendicular for } g_E) \end{aligned}$$

So in particular $V = TF$ and $H = TB_i$ over $B_i \times F \cong E|_{B_i}$. Define

$$\widetilde{g}_E = \begin{cases} g_E & \text{on } V \\ \pi^* g_B & \text{on } H \end{cases} \quad \text{and } V \perp H \text{ for } \widetilde{g}_E$$

$$v = \widetilde{\nabla} b + \varepsilon \textstyle\sum \rho_i \widetilde{\nabla} f_i \qquad (\widetilde{\nabla} = \text{ gradient for } \widetilde{g}_E)$$

Note that $\widetilde{g}_E = g_E$ on $\cup_i(\rho_i = 1)$.

Proof of ① and ②: ② is immediate.

$$\begin{aligned} db &= \widetilde{g}_E(\widetilde{\nabla} b, \bullet) \\ &= \widetilde{g}_E(\widetilde{\nabla} b, \text{project}_H \bullet) && \text{since } db = 0 \text{ on } V \\ &= (\pi^* g_B)(\text{project}_H \widetilde{\nabla} b, \text{project}_H \bullet) && \text{since } V \perp H \text{ for } \widetilde{g}_E \\ &= g_B(d\pi \widetilde{\nabla} b, d\pi \bullet) \\ \Rightarrow \quad & d\pi \cdot \widetilde{\nabla} b = \nabla b \\ \Rightarrow \quad & d\pi \cdot v = \nabla b && \text{since}^7 \ d\pi \widetilde{\nabla} f_i = 0 \\ \Rightarrow \quad & ① \checkmark \end{aligned}$$

Proof v is gradient-like: $dh(v) = |\widetilde{\nabla} b|^2 - \text{order}(\varepsilon) > 0$ outside $\rho_i = 1$, and on $\rho_i = 1$ v is the gradient of h by ② ✓.

We now construct a Morse-like complex for $-v$. Because of $(*)$, we define

$$C_* = MC_*(h) = MC_*(b) \otimes MC_*(f),$$

[6] indeed, the same proof holds: our index calculation shows that only the asymptotics of the linearization of the flow matter, and at the ends the flow is a Morse flow: $-v = -\nabla h$ by ②.

[7] $\widetilde{g}_E(\widetilde{\nabla} f_i, H) = df_i(H) = df(TB_i) = 0$ over B_i, and outside B_i have $\rho_i = 0$.

with differential

$$\begin{aligned} de &= \sum_{\dim V(e,e')=0,\ e\neq e'} \#V(e,e')\cdot e' \\ &= (d_0+d_1+d_2+\cdots)\,e \\ d_p e &= \sum_{|\pi e|-|\pi e'|=p,\ \dim V(e,e')=0} \#V(e,e')\cdot e'. \end{aligned}$$

Define the filtration:

$$F_p = \bigoplus_{|\pi e|\leq p,\ e\in \mathrm{Crit}\, h} \mathbb{Z}/2\cdot e.$$

Observe: $F_{\text{negative}}=0$, $F_{\dim B}=C_*$, $F_{p-1}\subset F_p$.

Crucial claim. F_p is a subcomplex: $dF_p\subset F_p$.
Proof. if $\exists -v$ traj u, then $\pi\circ u$ is a $-\nabla b$ traj.
$\Rightarrow |\pi e|-|\pi e'| = \dim W(\pi e,\pi e')\geq 0$
$\Rightarrow |\pi e'|\leq |\pi e|\leq p$ ✓

$\Rightarrow \boxed{E^0_{p,*-p}=F_p/F_{p-1}=MC_p(b)\otimes MC_{*-p}(f)}$ with $d=d^0$ on E^0.

Claim. $d_0=\partial_{\text{fibre}}$ counts $-\nabla f$ trajectories in the fibres.
Pf. $|\pi e|=|\pi e'|\Rightarrow W(\pi e,\pi e')=\emptyset$ unless $\pi e=\pi e'$, in which case $\pi\circ u$ =constant.✓

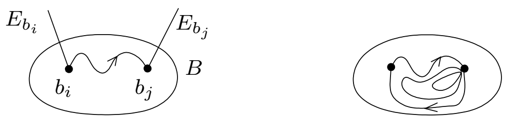
$\Rightarrow \boxed{E^1_{p,q}\cong MC_p(b)\otimes MH_q(f)}$

Warning: *this isomorphism is not canonical, because we made choices of trivializations. So let us be more precise:*

$$\begin{aligned} E^1_{p,q} &= \bigoplus_{b_i\in \mathrm{Crit}\, b} R_q(b_i) \\ R_q(b_i) &= MH_q(E_{b_i}, h|_{E_{b_i}} = b(b_i)+\varepsilon f_i|_{E_{b_i}}) \\ &\equiv MH_q(E_{b_i}, f_i|_{E_{b_i}}) \end{aligned}$$

and non-canonically $R_q(b_i)\cong MH_q(F,f)$ by using the choice of trivializations.

Using B simply connected. For simply connected B you can identify fibres by following a path in B, and if you change the path then you get a homotopic identification: hence the homology does not notice the change. Pictorial idea:[8]



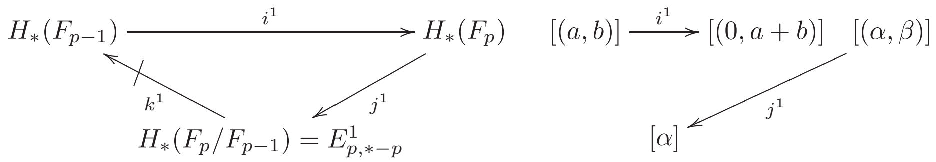
Write $F_p=C^p\oplus F_{p-1}$, where C^p is generated by the e with $|\pi e|=p$. Then

$$F_p = C^p\oplus F_{p-1} \xrightarrow{\ d=\left[\begin{smallmatrix} d_0 & 0\\ \partial' & \partial''\end{smallmatrix}\right]\ } C^p\oplus F_{p-1}$$

[8] on the right, compare parallel transports P_1,P_2 along two paths: homotope to a constant the loop that concatenates the two paths; obtain a chain homotopy between $P_2^{-1}\circ P_1$ and id.

where ∂', ∂'' are $d_1 + d_2 + \cdots$ composed with projection to C^p, F_{p-1} respectively.

$$\begin{array}{ccc} F_{p-1} = C^{p-1} \oplus F_{p-2} & \xrightarrow{i} & F_p = C^p \oplus F_{p-1} \\ (a,b) & \mapsto & (0, a+b) \end{array}$$



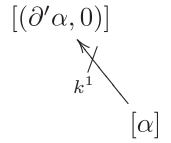
Recall k^1 is the boundary of the LES, so study the SES's:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (F_{p-1})_* & \longrightarrow & (F_p)_* & \longrightarrow & (F_p/F_{p-1})_* & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & (F_{p-1})_{*-1} & \longrightarrow & (F_p)_{*-1} & \longrightarrow & (F_p/F_{p-1})_{*-1} & \longrightarrow & 0 \end{array}$$

and diagram chase what happens to α:

$$\begin{array}{ccccccccc} & & & & (\alpha, 0) & \longrightarrow & \alpha & \longrightarrow & 0 \\ & & & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & (\partial'\alpha, 0) & \longrightarrow & d(\alpha,0) = (0, \partial'\alpha) & \longrightarrow & d_0\alpha = 0 & \longrightarrow & 0 \end{array}$$

so, by definition of the boundary k^1, in the triangle above we get



So $d^1[\alpha] = j^1k^1[\alpha] = [\partial'\alpha] = [d_1\alpha] \in H_{*-1}(F_{p-1}/F_{p-2})$ (here d_1 and ∂' agree since we quotient by F_{p-2}). Thus

$$\boxed{d^1[\alpha] = [d_1\alpha]}$$

We need to understand d_1:

$$d_1(b_i \otimes y) = \sum_{|b_j| = p-1, \text{ any } y' \in \operatorname{Crit} f} \#V_0(b_i \otimes y, b_j \otimes y') \cdot b_j \otimes y'$$

counts the 0-dimensional V spaces, and recall each $u \in V_0(b_i \otimes y, b_j \otimes y')$ lies over the $-\nabla b$ trajectory $\pi \circ u$ from b_i to b_j.

Note $d_1 : R(b_i) \to R(b_j)$ is a chain map (with respect to d_0) since:[9]

$$0 = d^2 = d_0^2 + (d_1 d_0 + d_0 d_1) + \cdots$$

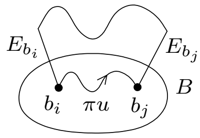
and $d_0^2 = \partial^2_{\text{fibre}} = 0$, hence $d_1 d_0 + d_0 d_1 = 0$. *(Exercise: you can also prove this last equality by a breaking analysis like in 7.4)*

Idea. d_1 counts trajectories which have an index drop in the base, so we expect the trajectories to be "constant" fibrewise. This makes sense if E is trivial (that is how we proved the Künneth thm), but otherwise the notion of "locally constant" depends on choices of local trivializations. So we need to keep track of choices.

[9] we break d^2 up according to the filtration, so each summand must vanish.

Claim. *Along a trivialization over the path $\pi \circ u$ the solutions u are continuation solutions of a Morse flow, and the count d_1 of such rigid solutions defines the same identification between $R(b_i)$ and $R(b_j)$ as the one induced on ordinary homology by parallel translation along any path joining b_i, b_j.*

Proof.



Pick a trivialization $\mathbb{R} \times F$ agreeing[10] with the given ones at b_i, b_j, so

$$h = \begin{cases} \varepsilon f + \text{constant} & \text{at } -\infty \\ \varepsilon f + \text{constant} & \text{at } +\infty \end{cases}$$

The count of isolated $-v$ flowlines in this trivialization (which project to the ∂_s flow in $\mathbb{R}$) then defines a map similar to a continuation map. Indeed, by covering the path πu by small charts, and extending the trivialization to these charts, we can homotope the metric $\widetilde{g}_E$ to make it a direct sum metric $g_B \oplus g_F$ (which it already is at the ends of the path πu), so that v is the gradient of a homotopy $h_s = b + \varepsilon f_s$, where f_s at the ends equals f. But now this homotopy can be homotoped to $h_s = b + \varepsilon f + c(s)$, where $c(s)$ only depends on s and at the ends equals the constants in the above expression for h at $\pm\infty$.

Observe that $\nabla(b + \varepsilon f + c(s)) = \nabla(b + \varepsilon f)$, so just as in the case of a constant hpy (6.3 (2)), one proves that $b + \varepsilon f + c(s)$ induces the identity continuation map. Hence, our original count of flowlines is chain homotopic to the identity. Hence on Morse homology it equals the identity. Thus the map agrees with the parallel transport map which defined the various trivializations (see footnote 10). ✓

Conclusion: Recover d_1 by finding the isolated $-\nabla b$ trajectories on B, and doing parallel transport in the fibres to get the map between the $R(b_i)$'s. More precisely:

$$\boxed{d_1 \text{ on } E^1 = \bigoplus_{b_i} R(b_i) \text{ can be identified with } \partial_{\text{base}} \text{ on } MC_*(b) \otimes MH_*(f)}$$

$$\Rightarrow E^2_{p,q} = MH_p(b) \otimes MH_q(f)$$

$$\Rightarrow \boxed{E^r_{p,q} \Rightarrow H_*(C_*, d)}$$

Finally, the last step of the proof of the Leray-Serre theorem, is:

$$H_*(C_*, d) \cong MH_*(h)$$

This is proved by a parametrized moduli space argument like in 6.3 (4): you homotope $-v$ to $-\widetilde{\nabla} h$. Note that $-v = -\widetilde{\nabla} h$ except in the regions where the $\rho_i \neq 0, 1$ which are small subsets of $B_i \setminus (\rho_i = 1)$. □

[10] Need to choose trivializations carefully: pick a trivialization $B_i \times F$ for some b_i, then parallel transport this over chosen paths to define the trivializations for the other $B_j \times F$'s. Finally we use the fact $\pi_1 B = 0$ to obtain the required trivialization with prescribed ends.