LECTURE 24.

PART III, MORSE HOMOLOGY, 2011

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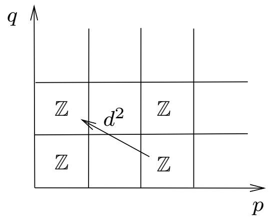
7.8. **Example of applying the Leray-Serre theorem.**
Let $E = \{v \in TS^2 : |v| = 1\}$ (the sphere bundle of TS^2).[1]

$$\begin{array}{ccc} S^1 \longrightarrow E & \qquad & E^2_{p,q} = MH_p(S^2) \otimes MH_q(S^1). \\ \quad\;\; \downarrow \pi & & \\ \quad\;\; S^2 & & \end{array}$$

Represent E^2 graphically as:

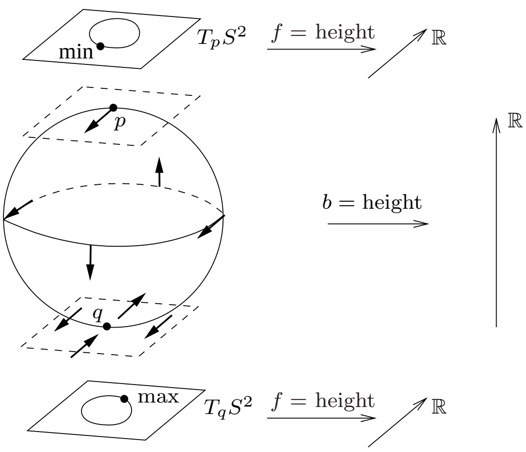


So we can already deduce:[a]

$$\begin{aligned} H^0(E) &= \mathbb{Z} \\ H^1(E) &= \mathbb{Z}/\operatorname{im} d^2 \\ H^2(E) &= \ker d^2 \\ H^3(E) &= \mathbb{Z} \end{aligned}$$

[a] here please take on trust that one can do Morse homology over $\mathbb{Z}$ by keeping track of orientation signs.

We take b = height function on S^2, and f = height function on the fibre S^1. To find d^2, we need to understand how parallel transport relates the critical points of index $1, 2$. Consider how a vector at the North pole p of S^2 gets parallel transported to the South pole q when moving along four great half-circles meeting at $90°$ at p.



We see that two[2] of the parallel transports of the vector point in the direction of the maximum in the S^1 fibre over q. Indeed, this shows that there are exactly two great half-circles from p to q such that the minimum in the fibre over p gets parallel

Date: May 3, 2011, © Alexander F. Ritter, Trinity College, Cambridge University.

[1] Secretly, one knows that $E \cong SO(3) \cong \mathbb{R}P^3$.

[2] secretly, this "two" is the Euler characteristic of the base S^2.

transported to the maximum in the fibre over q.
$\Rightarrow d^2 =$ multiplication by 2

$$\begin{array}{rcccccccc} \Rightarrow H_*(B) & = & \mathbb{Z} & \oplus & \mathbb{Z}/2 & \oplus & 0 & \oplus & \mathbb{Z} \\ * & = & 0 & & 1 & & 2 & & 3 \end{array}$$

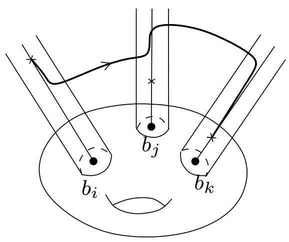
Rmk. *One can similarly do this for the higher dimensional case* $S^{n-1} \to S(TS^n) \to S^n$*. More generally, this method should in principle yield the Gysin sequence.*

8. Morse-Bott theory

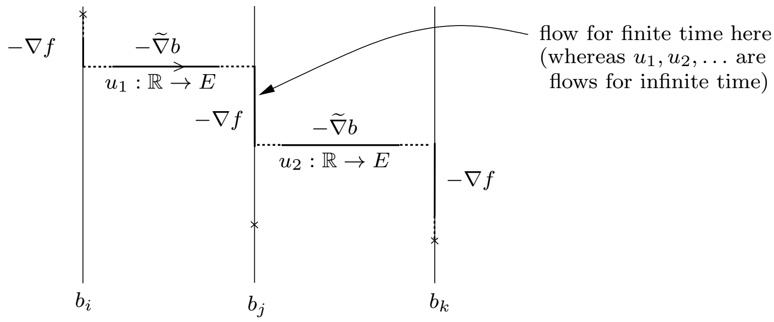
8.1. **Motivation.** *Question.* In the construction of the Leray-Serre spectral sequence, what happens if we let:

$$\begin{array}{l} \text{supports of } \rho_i \text{ shrink to } b_i \\ \text{and } \varepsilon \to 0. \end{array}$$

Answer. the trajectories become more vertical near the critical fibres, and more "horizontal" away from them:



So the trajectories converge to a combination of $-\nabla f$ flows along fibres and "quantum jumps" between the fibres given by $-\widetilde{\nabla} b$ flows:



8.2. **Morse-Bott functions.**
A smooth function $b : M \to \mathbb{R}$ is called *Morse-Bott* if

(1) $C = \operatorname{Crit} b = \bigsqcup_i C_i$ is a finite disjoint union of connected submfds $C_i \subset M$
(2) $\mathrm{Hess}_p b = D_p(db) : T_pM \to T_p^*M$ nondegenerate transversely to C_i, meaning

$$T_pC_i = \ker \mathrm{Hess}_p b \quad \forall p \in C_i.$$

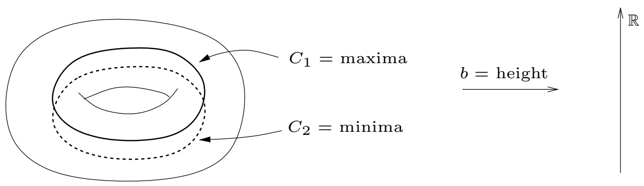
Equivalently: $\mathrm{Hess}_p b$ induces an invertible self-adjoint map on normal bdles

$$\mathrm{Hess}_p b : \nu_{C_i} \to \nu_{C_i}.$$

Examples.

(1) $b =$ *Morse*, $C = \{$*critical points*$\}$
(2) $b = 0$, $C = M$.

(3) *$b(x,y,z) = -x^2 + y^2$ and $C = z$-axis inside $\mathbb{R}^3$, but not $b = -x^3 + y^2$.*
(4) *a torus lying flat with the height function:*



(5) *Fibre bundle*
$$\begin{array}{ccc} F \to & E & \\ & \downarrow \pi & \\ & B & \xrightarrow{b} \mathbb{R} \end{array}$$
Suppose $b : B \to \mathbb{R}$ is Morse. Then $b \circ \pi : E \to \mathbb{R}$ is Morse-Bott with $C_i = \pi^{-1}(b_i)$ the fibres over the critical points b_i of b.

8.3. **Morse-Bott chain complex.** Choose auxiliary Morse functions

$$\boxed{f = \sqcup f_i : C = \sqcup C_i \to \mathbb{R}}$$

and a generic metric $g_C = \sqcup g_{C_i}$ on C. Write ∇f for the gradient of f w.r.t. g_C.

Def. *Define the grading of $p \in \mathrm{Crit}(f_i) \subset C_i$ by:*

$$\boxed{|p| = \mathrm{ind}_b(p) + \mathrm{ind}_f(p) = \mathrm{ind}\, C_i + \mathrm{ind}_f(p)}$$

Note that $\mathrm{ind}_b(p)$ is independent of $p \in C_i$ and is the index of $\mathrm{Hess}_p b : \nu_{C_i} \to \nu_{C_i}$.

Key Idea: *you are pretending that you perturbed b to $b + \varepsilon \sum \rho_i f_i$.*

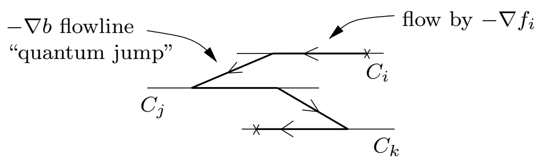
Example. *In example (3): $\mathrm{ind}_b(C) = 1$ because of the $-x^2$. For $f : C \to \mathbb{R}$, $(0,0,z) \mapsto -z^2$ get $|(0,0,0)| = -2$, which equals the index for $-x^2 + y^2 - \varepsilon z^2$.*

Def. *Define the* Morse-Bott complex *by*

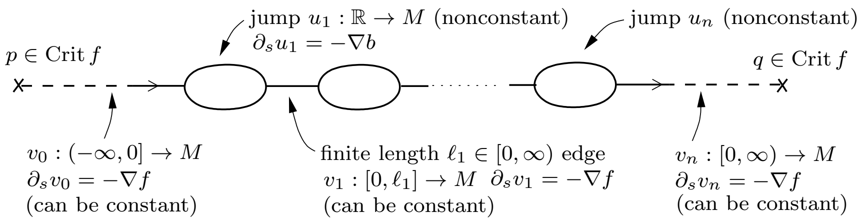
$$\boxed{\begin{aligned} BC_* &= \bigoplus_{p \in Crit\, f} \mathbb{Z}/2 \cdot p \\ &= \bigoplus_i MC_*(f_i)\,[\mathrm{ind}\, C_i] \end{aligned}}$$

The $[\mathrm{ind}\, C_i]$ is a shift in grading: $|p\,[\mathrm{ind}\, C_i]| = \mathrm{ind}_{f_i}(p) + \mathrm{ind}\, C_i$, so grading 0 becomes grading $\mathrm{ind}\, C_i$.

8.4. **Morse-Bott differential.** ∂ counts rigid Bott trajectories:



Useful Notation to summarize a Bott flowline:



Def. *The moduli space of Bott flowlines with $n \geq 1$ jumps is:*

$$W^n(p,q) = \{(u_1, \ldots, u_n; \ell_1, \ldots, \ell_{n-1}) : u_j \in W(p_j, q_j; b), p_j \neq q_j \in C \text{ such that } p_1 \in W^u(p,f), q_n \in W^s(q,f), \text{ and } q_j, p_{i+1} \text{ are connected by a finite time } -\nabla f \text{ flowline } v_j : [0, \ell_j] \to C, \ell_j \in [0,\infty)\}$$

$W^0(p,q) = W(p,q;f)$ *the moduli space of* $-\nabla f$ *flowlines* $\mathbb{R} \to C$.

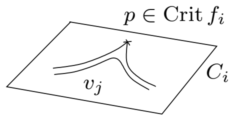
Def. *The moduli space of Bott trajectories is*

$$B(p,q) = \bigcup_{n \in \mathbb{N}} W^n(p,q)/\mathbb{R}^n,$$

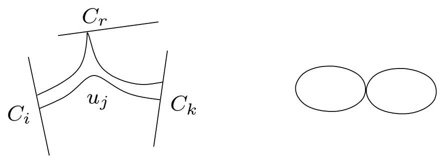
where $\mathbb{R}^n$ *acts by shifting the* s *coordinates in* $u_1, \ldots, u_n$.

8.5. **Breaking of Bott trajectories.**
Under C^0_{loc}-convergence, a Bott flowline can break in two ways:

(1) some $\ell_j \to \infty$ and therefore v_j breaks on some critical level set C_i:



(2) or some u_j breaks:



where on the right we indicated the abbreviated notation for the breaking.

The broken Bott flowlines in (2) for $W^n(p,q)$ are precisely the boundary points of $W^{n+1}(p,q)$ arising when $\ell_j = 0$. So when we compactify $B(p,q)$ we do not need to artificially add these limit broken Bott trajectories since they are already present. However, we still need to enlarge the topology so that it is recognized as a limit in the sense of (2). So we just artificially add the breakings of type (1):

$$\overline{B}(p,q) = B(p,q) \sqcup \bigsqcup_{n \geq 2} B(p,p_2) \times B(p_2,p_3) \times \cdots B(p_n,q).$$

$\partial \overline{B}(p,q) = \sqcup B(p,p_2) \times B(p_2,p_3) \times \cdots B(p_n,q)$ are called the *broken Bott trajectories*.

8.6. **Energy estimates for Bott trajectories.** Define the energy:

$$E(v_0, u_1, v_1, \ldots, u_n, v_n) = \sum \text{energies} = E(v_0) + E(u_1) + E(v_1) + \cdots + E(u_n) + E(v_n).$$

Lemma.

(1) *b decreases along a Bott trajectory*

(2) $\exists \delta > 0$ *such that to go from one* C_i *to another* C_j *a Bott trajectory must consume energy* $\geq \delta > 0$.
(3) *There are at most* $(f(p) - f(q))/\delta$ *jumps, so* $W^n(p,q) = \emptyset$ *for large* n.

Proof. b is constant along the v_i, and b decreases along u_i. (2) is proved like 3.3: $|\nabla b| > \delta > 0$ outside small nbhds of the C_i's, etc. and (3) follows from (2). □

For generic metrics g_M on M, g_C on C, one can prove the corresponding transversality, compactness and gluing results for $B(p,q)$ like we did for $\mathcal{M}(p,q)$, thus:

$B(p,q)$ smooth mfd
$\dim B(p,q) = \lvert p\rvert - \lvert q\rvert - 1$
$\overline{B}(p,q)$ is a compact mfd with corners

8.7. **Morse-Bott homology.** Recall

$$BC_* = \bigoplus_i MC_*(f_i)[\operatorname{ind} C_i].$$

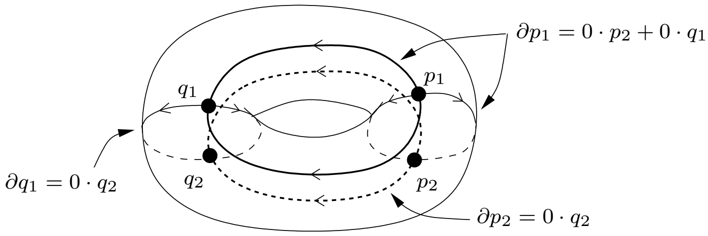
Define

$$\partial : BC_* \to BC_{*-1}$$
$$\partial p = \sum_{\dim B(p,q)=0,\ p\neq q} \#B(p,q) \cdot q.$$

The proof of $\partial^2 = 0$ follows just like for Morse homology from the results in 8.6. Hence we obain the Morse-Bott homology:

$$BH_*(b,f) = \frac{\ker \partial}{\operatorname{im} f}$$

Example. *In example (4) above, using height functions on the circles* C_1, C_2*:*



This shows $\partial = 0$*, so over* $\mathbb{Z}/2$*:*

$$\begin{array}{rcccccccc} BH_* = BC_* & = & \mathbb{Z}/2 & \oplus & (\mathbb{Z}/2)^2 & \oplus & \mathbb{Z}/2 & \cong & H_*(torus) \\ * & = & 0 & & 1 & & 2 & & \end{array}$$

8.8. **Invariance of Morse-Bott homology.** At this stage, one has to redo some of the work done for Morse homology:

(1) Build continuation maps,[3] when homotopying b, f_i, g_M, g_C.
(2) Prove invariance using continuation map properties.
(3) Invariance (2) implies[4] $BH_*(b,f) \cong BH_*(\text{Morse function } F, 0) \cong MH_*(F) \cong H_*(M)$, thus:

$$\boxed{BH_*(b,f) \cong H_*(M)}$$

[3] these are a little tricky, since homotopying b can change C drastically.
[4] Also, $BH_*(b,f) \cong BH_*(0, \text{Morse function } F \text{ on } M)$ works.

8.9. **Filtration by action.** For $a \in \mathbb{R}$,

$$F_a = F_a(BC_*) = \bigoplus_{y \in \mathrm{Crit}(f),\ f(y) \le a} \mathbb{Z}/2 \cdot y$$

Key observations: $dF_a \subset F_a$, $F_a = 0$ for $a \ll 0$, $F_a = BC_*$ for $a \gg 0$.

Now make it a discrete filtration, using the $\delta > 0$ of the energy estimates:

$$\cdots \subset F_0 \subset F_\delta \subset F_{2\delta} \subset \cdots \subset F_{p\cdot\delta} \subset \cdots$$

$\Rightarrow E^0_{p,*-p} = F_{(p+1)\delta}/F_{p\delta}$, and $d = d_0$ on E^0 since if you make a quantum jump, then you fall inside $F_{p\delta}$, so d only counts $-\nabla f$ trajectories in C.
$\Rightarrow E^1_{p,*-p} = \bigoplus_i MH_*(f_i)[\mathrm{ind}\, C_i]$
So we deduce:

Thm. *There exists a spectral sequence* $\boxed{E^1 = \bigoplus_i MH_*(f_i)[\mathrm{ind}\, C_i] \Rightarrow BH_*(b,f)}$

So there is also a spectral sequence $\boxed{\bigoplus_i H_*(C_i)[\mathrm{ind}\, C_i] \Rightarrow H_*(M)}$

Cor. *The Euler-characteristic* $\chi(M) = \sum (-1)^{\mathrm{ind}\, C_i} \chi(C_i)$.

Proof. This follows from the Theorem and from Lemma 7.3:
$\chi(M) = \chi(H_*(M)) = \chi(E^\infty) = \chi(E^1) = \chi\left(\oplus_i H_*(C_i)[\mathrm{ind}(C_i)]\right) = \sum (-1)^{\mathrm{ind}(C_i)} \chi(C_i)$. □

8.10. **Filtration by the index of b.** Make the following

Assumption. $B(p,q) = 0$ if $\mathrm{ind}_b(p) < \mathrm{ind}_b(q)$.

For example, this holds in example (5) above. Define

$$F_p = \bigoplus_{\mathrm{ind}_b C_i \le p} \ \bigoplus_{y \in \mathrm{Crit}(f_i)} \mathbb{Z}/2 \cdot y.$$

Then $d = d_0 + d_1 + d_2 + \cdots$, where

$$\begin{aligned} d_0 &= \text{counts } -\nabla f \text{ flowlines in } C \\ d_1 &= \text{allow one quantum jump} \\ d_2 &= \text{allow two quantum jumps} \\ &\cdots \end{aligned}$$

Thm. *There exists a spectral sequence of the same form as above.*

Example. *In example (5), we obtain the Leray-Serre spectral sequence. Use* $b : B \to \mathbb{R}$ *Morse on the base, and* $f_i : E_{b_i} \to \mathbb{R}$ *Morse on the fibres over* $b_i \in \mathrm{Crit}(b)$.

$$F_p = \bigoplus_{|b_i| \le p,\ y \in Crit\, f} \mathbb{Z}/2 \cdot y$$
$$E^1 = \bigoplus_{b_i \in \mathrm{Crit}(b)} MH_*(f_i)\,[\,|b_i|\,]$$

For $\pi_1(B) = 0$*, get* $d^1 = \partial_{\mathrm{base}}$. *So*

$$E^2_{p,q} = MH_p(B) \otimes MH_q(F) \Rightarrow H_*(E).$$

Note this has the enormous advantage that we do not have to construct a special metric and a pseudo-gradient vector field.

9. Where to go from here

I recommend three interesting survey papers:

- Michael Hutchings, *Lecture Notes on Morse homology.*

  This is available online. It is a very elegant treatment of many interesting topics. It covers parts of this course, but sometimes using a different approach (some proofs in Morse homology can be simplified if one uses smooth dependence of ODE's on initial conditions, but unfortunately these proofs do not generalize to Floer theory so we avoided this approach).

- Dietmar Salamon, *Lectures on Floer homology.*

  This is available online. It is the best place to learn the basics of Floer homology. Always short and to the point, which is wonderful.

- Kenji Fukaya, *Morse homotopy, A^∞-category and Floer homologies.*

  Available online (the fonts are a little strange). This is excellent to get a feel for the ideas involved in Floer theory.

For research directions on more advanced topics, I recommend three books:

- Dusa McDuff and Dietmar Salamon, *J-Holomorphic Curves and Quantum Cohomology*, 1994 (not the similarly called 2004 version).

  This is a great book and is very readable.

- Paul Seidel, *Fukaya categories and Picard-Lefschetz theory.*

  This is a very advanced book. It is the key reference for A^∞-algebras, Lagrangian Floer homology, Lefschetz fibrations, Fukaya categories. This is useful if you become a mathematician in the area of symplectic topology.

- Peter Kronheimer and Tomasz Mrowka, *Monopoles and Three-Manifolds.*

  This is a very detailed treatment of Seiberg-Witten Floer homology. It always motivates ideas using Morse homology, which is a great approach.