Morse Homology, with a view towards Floer theory (TCC course 2024)

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(google me to find my webpage for notes)

- Old Cambridge part III notes for Morse Homology
- Book by Audin & Damian (Springer) on Morse Theory / Homology

Today: Overview (some proofs later in course)

A nice class of functions

M closed smooth manifold of $\dim = m$

closed: compact without boundary

manifold: assume some differential geometry e.g. nice book by Guillemin & Pollack or my webpage course on Lie groups: first few pages has crash course on mfds.

smooth function

$$f : M \to \mathbb{R}$$

Locally:

$$f : \mathbb{R}^m \to \mathbb{R} \quad \text{near } p = 0$$

assume convergent Taylor series

$$f(x) = f(0) + \sum \partial_{x_i} f(0) \cdot x_i + \frac{1}{2!} \sum \partial_{x_i} \partial_{x_j} f(0)\, x_i x_j + \ldots$$

$$= f(0) + \underline{df_0} \cdot x + \frac{1}{2} x^T \underline{\mathrm{Hess}_0(f)}\, x + \ldots$$

Want: . few $p \in M$ that are critical i.e. $df_p = 0$

. if critical point p, want matrix nondegenerate i.e. $\det \neq 0$

Def f is a Morse function if all critical points are nondegenerate.

Fact $\Rightarrow$ critical points are isolated, so finite # crit. pts

$\Rightarrow$ locally $\exists$ coordinates so that (near $p \in \mathrm{crit} f$):

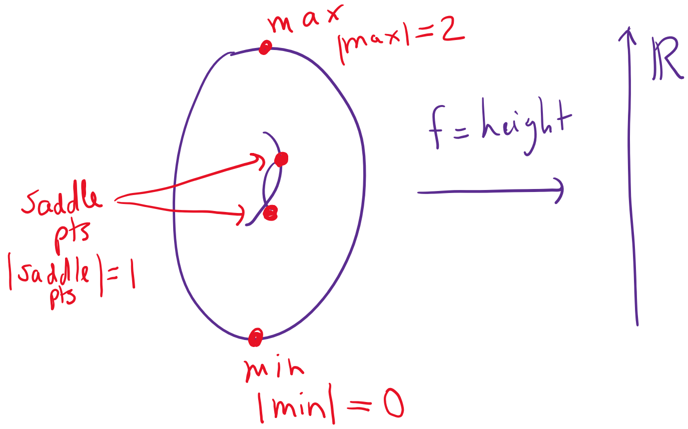
$$f = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_m^2$$

$p = 0$

$$\mathrm{Hess} = \begin{pmatrix} -1 & & & & & 0 \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}$$

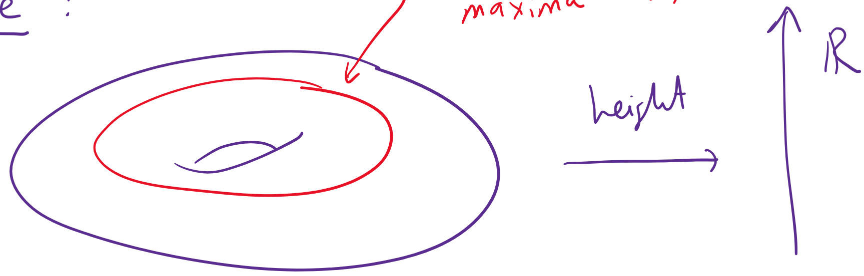
$k =$ # these negative evals of $\mathrm{Hess}_p(f)$ called <u>index</u> of p $\leftarrow$ $\boxed{|p| = k}$

Example $T = \text{torus} = S^1 \times S^1$



Morse

not Morse:

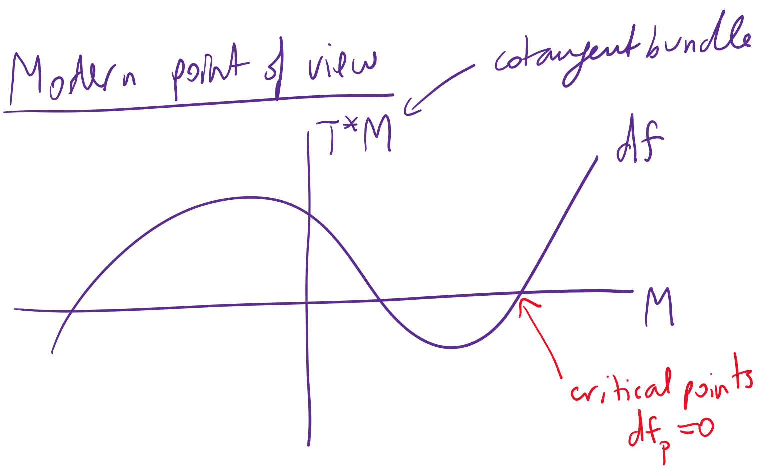


$\Rightarrow$ crit(f) not isolated

$\Rightarrow$ not Morse

indeed locally $\mathrm{Hess} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$

Modern point of view

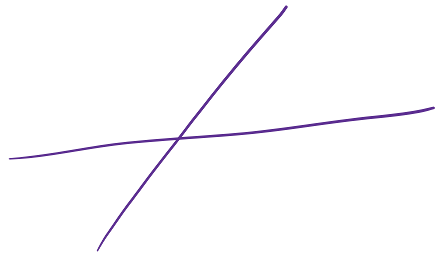


f Morse

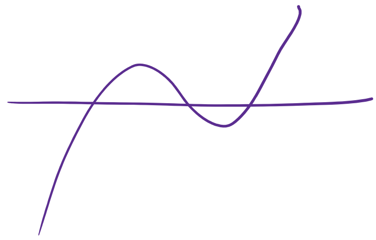
$\Updownarrow$

intersections between graph(df) and zero section in T^*M are <u>transverse</u>

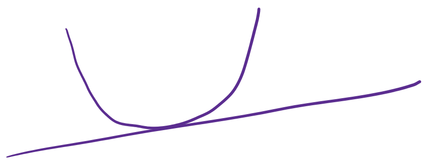
Idea of Transversality in $\mathbb{R}^2$:



or



not transverse :



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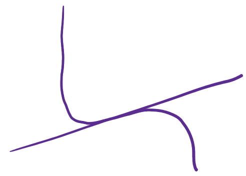
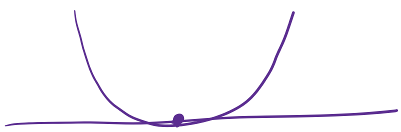
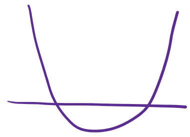


in geometry : generically things are transverse

idea



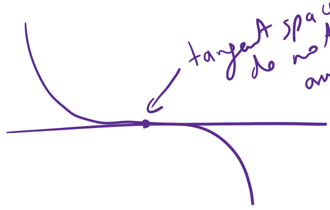
perturb $\longrightarrow$



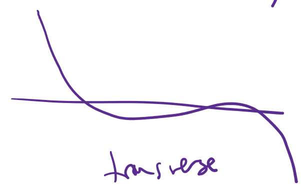
counts finite but wrong

(expect 2 or 0 intersections with signs get 0 count)

tangent spaces together do not span ambient tang. space



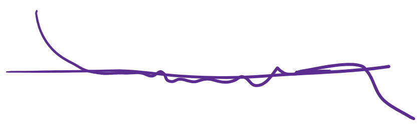
$\longrightarrow$ perturb



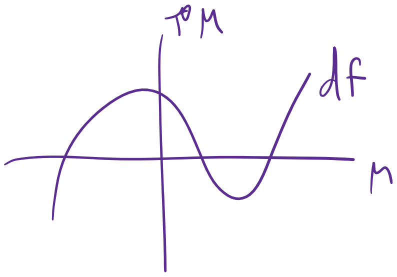
transverse

(or )

not transverse



∞ # intersections cannot count



Fact

generic $f \implies df \pitchfork$ 0-section

$\nwarrow$ transverse

$\implies f$ Morse

so "almost all functions are Morse!"

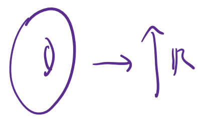
$\nwarrow$ in course will explain

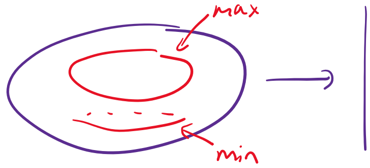
Fact Any manifold can be embedded inside $\mathbb{R}^N$ $\leftarrow$ some N

Fact Almost any "height function" is Morse for $M \subseteq \mathbb{R}^N$

$\nwarrow$ linear functional $\mathbb{R}^N \to \mathbb{R}$ (restricted to M)

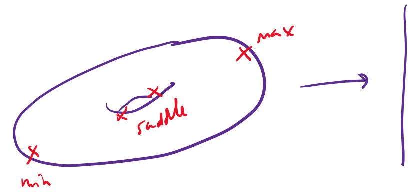
e.g. torus $T \subseteq \mathbb{R}^3$





not generic

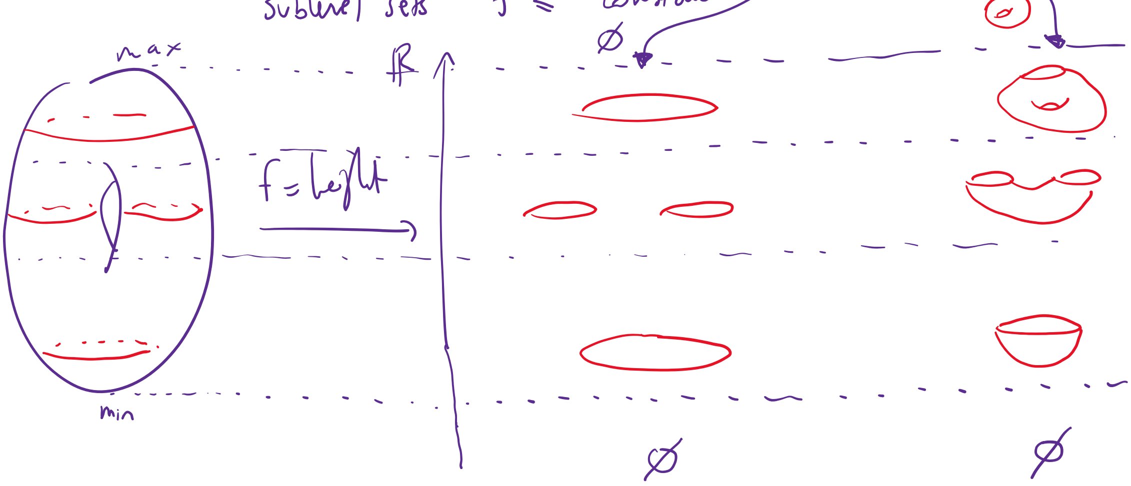
⇝ perturb in functional so generic, same as slightly tilting torus



What do Morse functions tell us about homology of M?

Look at level sets $f = \text{constant}$

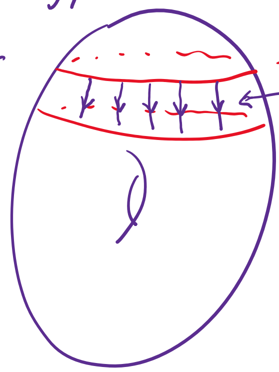
sublevel sets $f \leq \text{constant}$



Topology changes when pass through critical values ↖ $f(\text{critical point})$

Topology does not change if don't cross critical value:

no critical points inside $\{$ $r \le f \le r+\varepsilon$

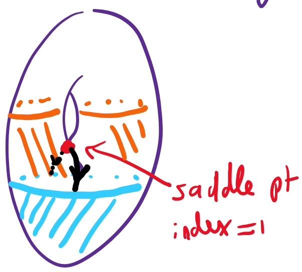


flow (for time ε) by vector field in direction of $-\nabla f$

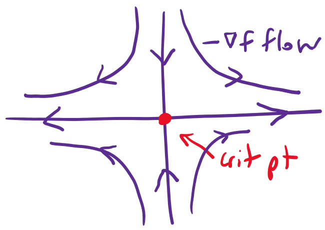
$$\frac{-\nabla f}{\|\nabla f\|^2} \leftarrow \text{rescaling ensures level sets are preserved by flow}$$

(choice of Riemannian metric to define ∇f)

<u>Fact</u> when pass critical value, topology changes by attaching a k-cell for each critical point of index k



local coords:

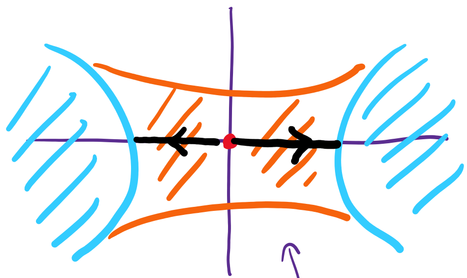


$f(x,y) = c - x^2 + y^2$

$c = f(p)$

$$\mathrm{Hess}_p = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

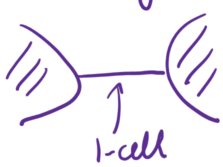
index = #(negative evals) = 1



"1-handle" — index of p

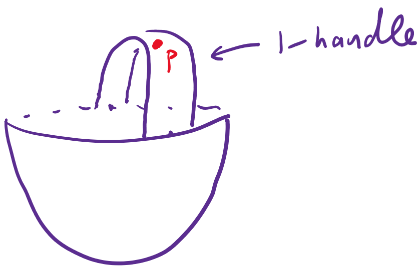
attaching a fattening of 1-dim disc $\mathbb{D}^1$ ← "one-cell"

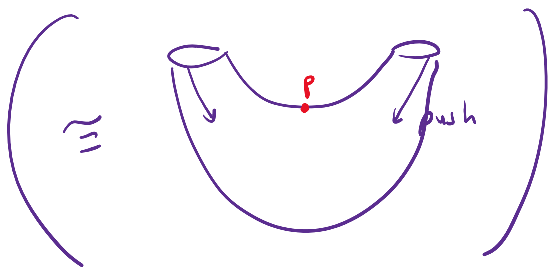
hpy equivalent to



<u>Fact</u> can reconstruct the manifold up to homotopy equivalence

<u>Also up to homeomorphism:</u>





<u>Warning</u>: don't quite know mfd up to diffeomorphism just from this information: need to know the attaching maps.

<u>Example</u> <u>Fact</u> f Morse, only 2 critical points $\Longrightarrow M \underset{\text{homeo}}{\cong} S^m$ ← m-dim sphere

however : may not be diffeomorphic to standard sphere

Milnor : $\exists$ such examples in $m=7$ where get mfd M homeo to S^7 but not diffeo

"exotic spheres"!

<u>idea</u> : M compact $\Longrightarrow \exists$ max, min so those are the 2 crit pts

max: index $= m$; min: index $= 0$

} ← attachment of m-cell i.e.

← topology does not change

min ← locally $f = f(\text{min}) + \sum x_i^2$

so sublevel sets are discs nearby (m-dim)

attaching map $S^{m-1} \xrightarrow{a} S^{m-1}$

$\theta \longmapsto a(\theta)$

($\mathbb{D}^m$ (r, θ)

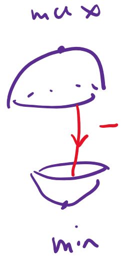
origin $0 \in \mathbb{D}^n$ is "$(0, \underset{\theta}{\text{any}})$"

extend continuously to top

$a(r,\theta) = a(\theta) \cdot r$

cannot always extend over North smoothly, causes exotic examples)

<u>e.g.</u>
<u>S^2</u>



← locally $f = f(\text{max}) - x_1^2 - x_2^2$

$-\frac{\nabla f}{\|\nabla f\|^2}$ flow produces diffeo $S^1 \to S^1$ (in particular degree 1)

(why not -1? idea: $-\nabla f$ is normal to boundaries of two discs)

← $f = f(\text{min}) + y_1^2 + y_2^2$ locally

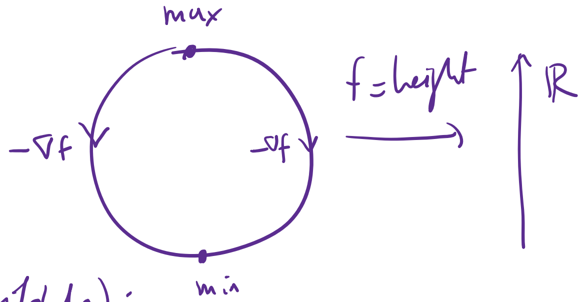
<u>Technical Rmk</u>

The attaching maps we get in our setup are always homeomorphisms onto the image. The homeomorphism type of the topological manifold we get only depends on the isotopy class of the attaching maps.

(continuous family of homeomorphisms)

Homology?

Example: $M = S^1$



Unstable cells (or unstable manifolds):

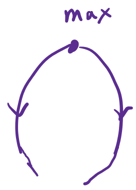
$$\mathcal{U}(p) = \{\text{points flowing down from } p \text{, including } p\}$$

p: critical pt

flowing down: via $-\nabla f$

$$\left\{ x \in M : \lim_{t \to -\infty} \text{flow}_t^{-\nabla f}(x) = p \right\}$$

W_p^- or W_p^u

$\mathcal{U}(\text{max}) =$ 

(open) 1-cell

$\mathcal{U}(\text{min}) = \cdot$ min

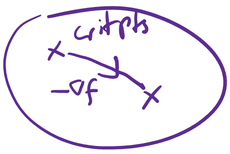
0-cell

cellular boundary: $\partial\, \mathcal{U}(\text{max}) = \underbrace{\mathcal{U}(\text{min}) - \mathcal{U}(\text{min})}_{=0}$

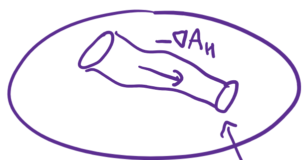
$$\Rightarrow \quad H_*^{\text{cell}}(S^1) = \begin{cases} \mathbb{Z} \cdot \text{min} & * = 0 \\ \mathbb{Z} \cdot \text{max} & * = 1 \end{cases}$$

Issues: in ∞ dim setting often flow $-\nabla f$ is not well-defined for arbitrary time

e.g. Floer theory



Morse
"particles moving"



"closed strings"

Hamiltonian orbits

flow ill-defined (of functional A_H)

How avoid using $\mathcal{U}(p)$?

Idea Witten, Floer ~1980

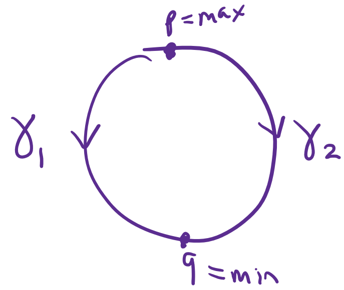
Instead consider <u>moduli space of flowlines</u> between 2 crit pts

$$\mathcal{M}(p,q) = \left\{ \begin{matrix} -\nabla f \text{ flowlines} \\ \text{from } p \text{ to } q \end{matrix} \right\} \Big/ \text{reparametrisation}$$

$$\left(\begin{matrix} \gamma : \mathbb{R} \to M, \ \gamma'(t) = -\nabla f \\ \gamma(-\infty) = p, \ \gamma(+\infty) = q \end{matrix} \right) \qquad \gamma(\cdot + \text{constant})$$

"time-shift"

E.g. $M = S^1$



$$\mathcal{M}(p,q) = \{\gamma_1, \gamma_2\}$$

Morse complex:

index of crit pt

avoid talking about orientation signs

$MC_*(f) =$ freely generated over $\mathbb{Z}/2$ by critical pts of Morse f

for S^1: $\qquad = \mathbb{Z}/2 \cdot p \oplus \mathbb{Z}/2 \cdot q$

$|p| = 1 \qquad |q| = 0$

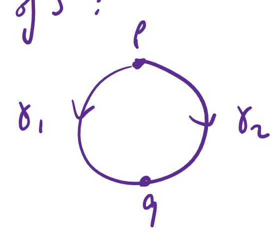
Differential

$$d : MC_k \longrightarrow MC_{k-1}$$

$$dp = \sum \left(\# \mathcal{M}(p,q) \right) \cdot q$$

count # elements

case of S^1:



$$\begin{matrix} \mathbb{Z}/2 \cdot p \\ \downarrow dp = 2q = 0 \\ \mathbb{Z}/2 \cdot q \end{matrix} \qquad \text{(working modulo 2)}$$

$\# \{\gamma_1, \gamma_2\}$

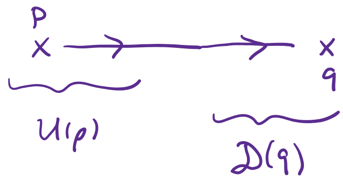
and $dq = 0$

$$\Rightarrow H^{Morse}_*(S^1) = \begin{cases} \mathbb{Z}/2 \cdot p & * = 1 \\ \mathbb{Z}/2 \cdot q & * = 0 \end{cases}$$

Why $\mathcal{M}(p,q)$ better than $\mathcal{U}(p)$?

finite dim case: $$\mathcal{M}(p,q) = \left(\mathcal{U}(p) \cap \mathcal{D}(q)\right) / \text{reparam}$$

stable mfd
pts flowing into q
including q



in ∞ dims: $\mathcal{U}(p)$, $\mathcal{D}(q)$ are usually ∞ dim'l, even when well-defined

Rmk: Intersection theory in ∞ dims is dangerous: e.g. think about $(a_0, 0, a_1, 0, a_2, 0, \ldots)$ $\xrightarrow{\text{isotope}}$ $(a_0, 0, 0, a_1, 0, 0, a_2, \ldots)$

miracle: one can ensure
$$\mathcal{U}(p) \cap \mathcal{D}(q)$$
are finite dim'l! So $\mathcal{M}(p,q)$ are nicer spaces than $\mathcal{U}(p)$
(subject to achieving transversality – generic setup)

<u>Trick</u> auxiliary choices: g Riemannian metric

determines ∇f (and hence the flow of $-\nabla f$)

$$df = \underline{g}(\nabla f, \cdot) \quad \text{defines v.field } \nabla f$$

need g to define ∇f

<u>Fact</u> generic g (or "perturb" a given g) $\Longrightarrow$ $\mathcal{M}(p,q)$ are finite dim'l mfds (indeed $\mathcal{U}(p) \pitchfork \mathcal{D}(q)$)

TRANSVERSALITY THEOREM

and $$\dim \mathcal{M}(p,q) = |p| - |q| \underbrace{- 1}_{\mathbb{R}\text{-reparametrisn}}$$

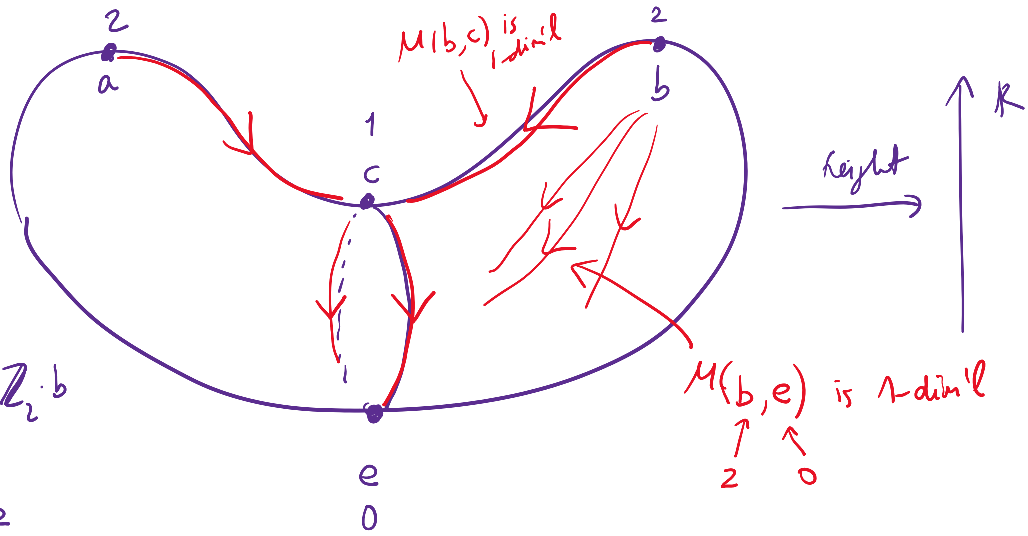
in particular if index difference $= 1$ get 0-dim'l moduli space $\Rightarrow$ can count it, if finite

Second Key ingredient:

<u>COMPACTNESS THEOREM</u> $\mathcal{M}(p,q)$ is compact if index difference $= 1$.

Example

$M = $ hot dog $\cong S^2$



$MC_2 = \mathbb{Z}_2 \cdot a \oplus \mathbb{Z}_2 \cdot b$

$MC_1 = \mathbb{Z}_2 \cdot c$

$MC_0 = \mathbb{Z}_2 \cdot e$

$$\left.\begin{array}{l} d(a) = c \qquad\qquad d(b) = c \\ d(c) = 2e = 0 \pmod 2 \\ d(e) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} MH_2 = \mathbb{Z}_2 \cdot (a-b) \\ MH_1 = 0 \leftarrow \text{c is also bdry} \\ MH_0 = \mathbb{Z}_2 \cdot e \end{array}$$

$\Rightarrow MH_* \cong H_*(S^2)$

singular, cellular, etc.

Thm $\forall$ Morse f $\qquad MH_*(f) \cong H_*(M)$

Application

$\dim_{\mathbb{Z}/2}$ in our case

$$\begin{aligned} \# \operatorname{Crit}(f) &= \operatorname{rank} MC_* \\ &\geq \operatorname{rank} MH_* \\ &= \operatorname{rank} H_* \end{aligned}$$

e.g. on torus : $\operatorname{rank} H_* = 4 \quad (= 1 + 2 + 1)$

Morse functions on torus must have ≥ 4 crit. pts