CONTINUE OVERVIEW

Notes on my website (Google Alex Ritter Oxford)

Last time:

M closed mfd, pick Riemannian metric g

$f : M \to \mathbb{R}$ Morse function

Morse cx: $MC_* = \bigoplus_{x \in \mathrm{Crit} f} \mathbb{Z}/2 \cdot x$

$$d(p) = \sum_{\underbrace{|p|-|q|-1=0}_{\dim \mathcal{M}(p,q)=0}} \# \underbrace{\mathcal{M}(p,q)}_{\text{moduli space of flowlines}} \cdot q \qquad \text{and extend } \mathbb{Z}/2\text{-linearly}$$

$$\left\{ \begin{array}{l} \gamma : \mathbb{R} \to M : \ \gamma'(t) = -\nabla f \\ \qquad \gamma(t) \to p \ \text{ as } \ t \to -\infty \\ \qquad\qquad\ \ \, q \qquad\ \ t \to +\infty \end{array} \right\} \Big/ \sim \qquad g(\cdot, \nabla f) = df$$

$\gamma(\cdot) \sim \gamma(\cdot + \text{constant})$ $\mathbb{R}$-reparametris^n.

$MH_* = \mathrm{Ker}\, d / \mathrm{Im}\, d$

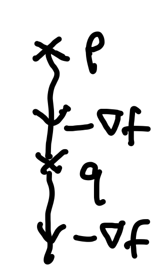
Compactness theorem:

generic $g \Rightarrow \mathcal{M}(p,q)$ is compact 0-dim mfd when $|p|-|q|-1=0$

Why is it a complex, so $d^2=0$?

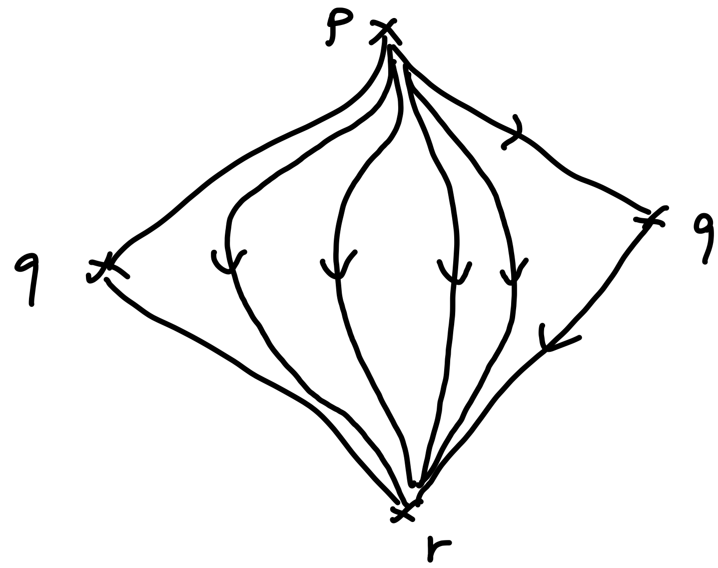
Idea of proof: $|p|-|r|-1 = 1$ so 1-family of trajectories

$$\begin{aligned} d^2 p &= d\left(\sum \# \mathcal{M}(p,q) \cdot q\right) \\ &= \sum \# \mathcal{M}(p,q) \cdot dq \\ &= \sum_{\substack{q,r \\ |p|-|r|-1=1}} \underbrace{\# \mathcal{M}(p,q) \cdot \# \mathcal{M}(q,r)}_{\text{"broken flowlines"}}\, r \end{aligned}$$



$d^2 = 0 \iff$ once-broken flowlines arise in pairs

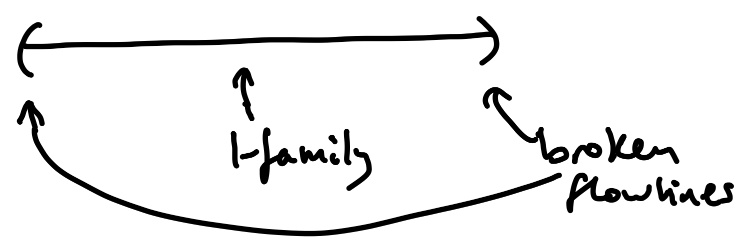
idea:



one shows

$\mathcal{M}(p,r)$ = open 1-mfd that can be compactified by 1-broken flowlines

above:



compactified $\overline{\mathcal{M}}(p,r)$ = compact 1-mfd with bdry
$= \mathcal{M}(p,r) \cup \partial \overline{\mathcal{M}(p,r)}$

COMPACTNESS THEOREM 2

$\overline{\mathcal{M}(p,r)}$ = disjoint union of circles and closed intervals

$\partial \overline{\mathcal{M}(p,r)}$ = even # of points

broken flowlines $= \# \partial \overline{\mathcal{M}(p,r)} = 0 \mod 2$

$\Rightarrow d^2 = 0$

0-dim case

1-dim case

Overview of why $\mathcal{M}(p,q)$, $\mathcal{M}(p,r)$ are manifolds

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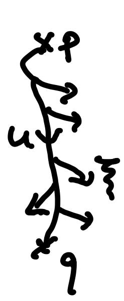
Functional analysis: build a "Banach" vector bundle



$\{$smooth vector fields along $u\}$

$\downarrow$

$\{$smooth paths $u : \mathbb{R} \to M$
$u(t) \to p, q$ as $t \to -\infty, \infty\}$

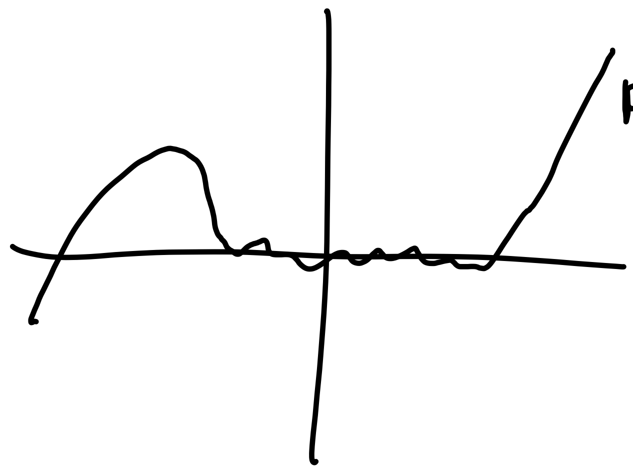
section
$F(u) = \partial_t u - \nabla f$

so:
$u \in F^{-1}(0) \iff u$ is $-\nabla f$ flowline

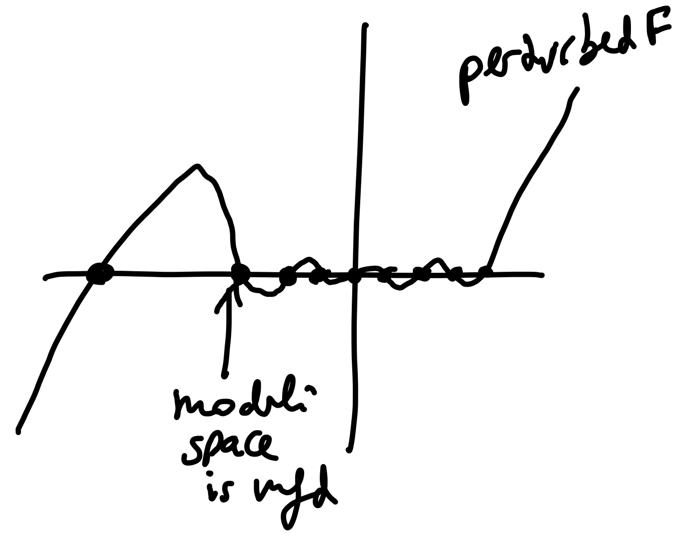
$\mathcal{M}(p,q)$ before quotienting by $\mathbb{R}$-reparam. $= \begin{pmatrix} \text{0-section of} \\ \text{Banach} \\ \text{vector bundle} \end{pmatrix} \cap \begin{pmatrix} \text{section } F \\ F(u)=\partial_t u - \nabla f \end{pmatrix}$

is this a <u>transverse</u> intersection?

Key: shows generic g this is transverse:



$\xrightarrow{\text{perturb } g}$



called TRANSVERSALITY THEOREM

Last time: <u>Fact:</u> M closed mfd $\Rightarrow$ $MH_*(f) \cong H_*(M)$ $\forall$ Morse f

<u>Classical results in algebraic topology revisited</u>

Poincaré duality : $H_*(M) \cong H^{n-*}(M)$ $\quad n = \dim M$
over $\mathbb{Z}/2$
(or over $\mathbb{Z}$ if M orientable)

$MC_*(f) \xrightarrow{\text{identity}} MC^{n-*}(-f)$ ← still Morse

$\operatorname{Crit} f = \operatorname{Crit}(-f)$
$\operatorname{Hess}_p f$ changes sign so
$|p| = \#$neg. evals
for $-f$: $\#$pos. evals $= n - |p|$

d counts
$\gamma'(t) = -\nabla f$
$\gamma \to p,q$ at $-\infty, \infty$

$\xrightarrow{\text{associate}}$

$\tilde{\gamma}(t) = \gamma(-t)$
$\tilde{\gamma}'(t) = -\gamma'(-t)$
$= -(-\nabla f)$
$= -\nabla(-f)$
$\tilde{\gamma} \to \underline{\underline{q,p}}$ at $-\infty, \infty$ ← switches roles of p,q

"cochain differential"
$d(q) = \sum_{\dim = 0} \# \underline{\underline{\mathcal{M}(p,q)}} \cdot p$ ← now counting <u>incoming</u> flowlines

∴ Poincaré duality is just $f \leftrightarrow -f$ or intuitively "look at mfd upside down"

<u>Künneth isomorphism</u> (thm)

$$H_*(M_1 \times M_2) = H_*(M_1) \otimes H_*(M_2)$$

$$\cong$$

$$HM_*(f_1 + f_2)$$

f_1: Morse $M_1 \to \mathbb{R}$; f_2: Morse $M_2 \to \mathbb{R}$

$\underbrace{\qquad\qquad}_{\text{Morse on } M_1 \times M_2 \to \mathbb{R}}$

$\mathrm{Crit}(f_1 + f_2) = (p_1, p_2) \in \mathrm{Crit} f_1 \times \mathrm{Crit} f_2$

$(\gamma_1, \gamma_2) : \mathbb{R} \to M_1 \times M_2$

$$\begin{cases} \gamma_1' = -\nabla f_1 \\ \gamma_2' = -\nabla f_2 \end{cases}$$

d only counts <u>isolated</u> (or <u>rigid</u>) flowlines mod $\mathbb{R}$ reparam

if <u>both</u> γ_1, γ_2 were not constant then get $\mathbb{R}^2$-reparam freedom:

$$(\gamma_1(\cdot + c_1), \gamma_2(\cdot + c_2))$$

$\Rightarrow$ either γ_1 or γ_2 is constant

$\Rightarrow$ $d = d_1 \otimes id + id \otimes d_2$ (mod 2)

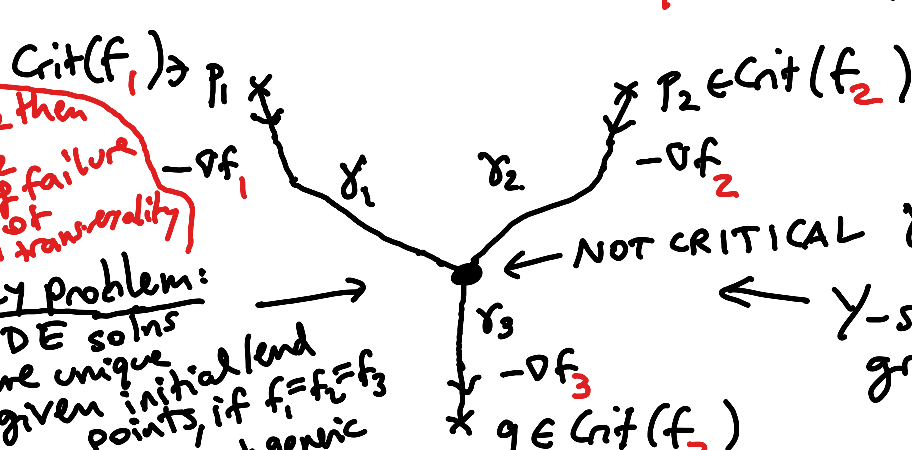
$\Rightarrow$ $MC_*(f_1 + f_2) \equiv MC_*(f_1) \otimes MC_*(f_2)$ ∎

<u>Cup product</u>

$$H^a(M) \otimes H^b(M) \xrightarrow{\smile} H^{a+b}(M)$$

pick generic Morse functions $f_1, f_2, f_3 : M \to \mathbb{R}$

$$MH^a(f_1) \otimes MH^b(f_2) \overset{?}{\dashrightarrow} MH^{a+b}(f_3)$$



if $f_1 = f_2$ then $\gamma_1 \equiv \gamma_2$ causing failure of transversality

Key problem: ODE solns are unique given initial/end points, if $f_1 = f_2 = f_3$ not generic

$p_1 \otimes p_2 \longmapsto \sum \# \mathcal{M} \cdot q$

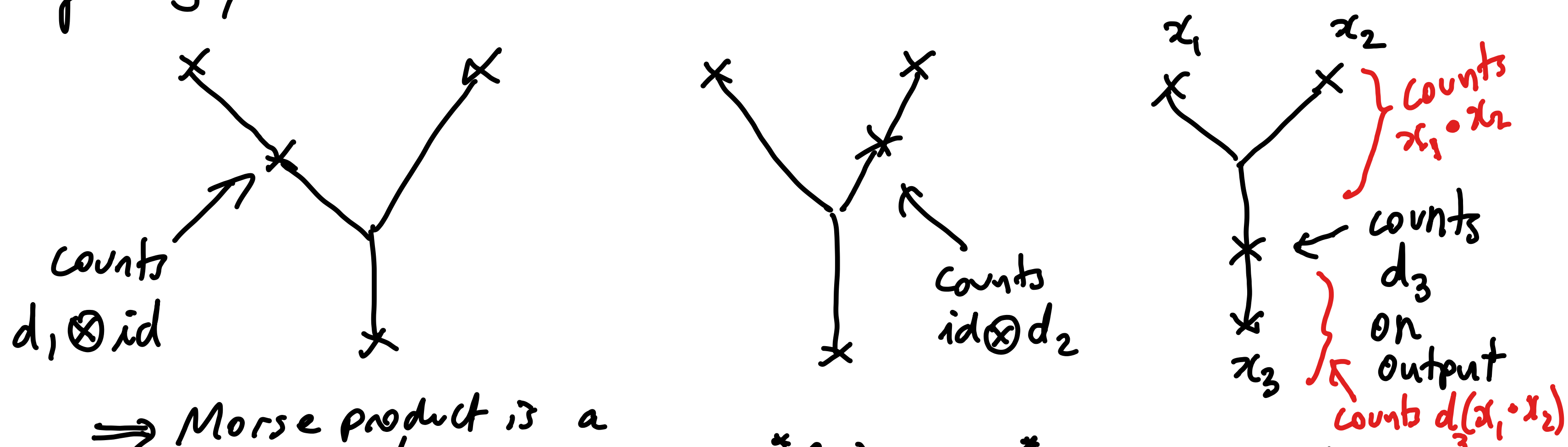
some moduli space $\mathcal{M}(p_1, p_2; q)$

$\gamma_1(0) = \gamma_2(0) = \gamma_3(0)$

Y-shaped graph $\leftarrow$
$(-\infty, 0] \xrightarrow{\gamma_1} M$
$(-\infty, 0] \xrightarrow{\gamma_2} M$
$[0, +\infty) \xrightarrow{\gamma_3} M$

(0 is reached in finite time)

in 1-family, can break as:



$\Rightarrow$ Morse product is a chain map $MC^*(f_1) \otimes MC^*(f_2) \xrightarrow{\bullet} MC^*(f_3)$

counts Y-shaped flowlines

analogue in alg. top. of

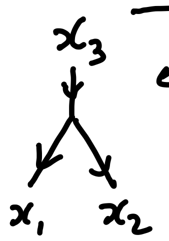
$$d(c_1 \cup c_2) = dc_1 \cup c_2 \pm c_1 \cup dc_2$$

(ensures get $\cup$ on H^*)

in our case

$$d_3(x_1 \bullet x_2) = (d_1 x_1) \bullet x_2 \pm x_1 \bullet (d_2 x_2)$$

Why grading a+b on output?



cohomological conventions:

TRANSVERSALITY THM: if f_1, f_2, f_3 generic then $\mathcal{M}$ ( ) is smooth mfd of that dim

$$\dim \mathcal{M}(x_3; x_1, x_2) = |x_3| - |x_1| - |x_2|$$

$$x_1 \bullet x_2 := \sum_{\dim \mathcal{M}=0} \# \mathcal{M}(x_3; x_1, x_2) \cdot x_3$$

for critical x_1, x_2, x_3 then extend bilinearly

$\in MC^*(f_1)$ $\quad \in MC^*(f_2)$ $\quad \in MC^*(f_3)$

$\mathcal{M}$ compact 0-mfd so get finite count and $|x_3| = |x_1| + |x_2| = a+b$

Notice again: we only count "rigid" solutions.
"isolated"

Intuition of why cup-product:

$$MH_*(f) \xrightarrow{\cong} H_*(M)$$

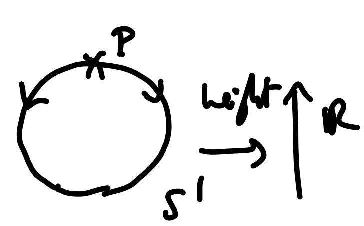
$$p \longmapsto \text{"}[\mathcal{U}(p)]\text{"}$$

$\cap$
crit(f)

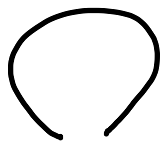
unstable manifold has dim $= |p|$

paper by M. Schwarz making this rigorous
Idea "pseudocycle"
allow ∂(cycle) to be covered by submfds of dim 2 less than cycle
e.g. $\partial(S^2 \setminus pt) \subseteq S^2$

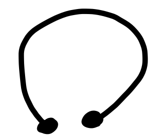
$\mathcal{U}(p)$ is open and not-properly-embedded manifold

e.g. 

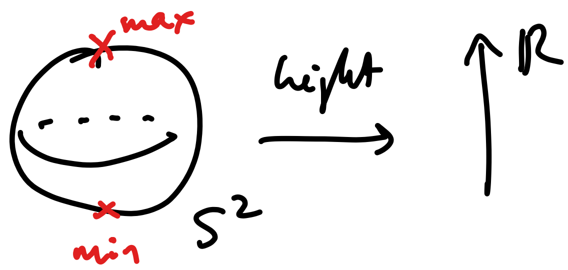
$\mathcal{U}(p) =$  open 1-cell, $\mathbb{R} \hookrightarrow S^1$ not proper map



pseudocycle compactified to



honest 1-cycle



$$MC_*(\text{height}) \longrightarrow H_*(S^2)$$

$$\text{min} \longmapsto [\mathcal{U}(\text{min})] = [pt] \in H_0(S^2)$$

$$\text{max} \longmapsto [\mathcal{U}(\text{max})] = [\underbrace{S^2 \setminus pt}_{\substack{\uparrow \\ \text{pseudocycle}}}] \in H_2(S^2)$$

is $[S^2]$ in ordinary homology

think of $\mathbb{R}^2 \to S^2$ as compactifying to
$D^2 \to S^2$
$\cup$
$\partial D^2 = S^1 \to pt$

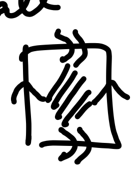
covered by submfds of dim 2 less than degree of $|p| = 2$
indeed just one point

Exercise Make sense of
$$MH_*(f) \longrightarrow H_*(T^2)$$
for

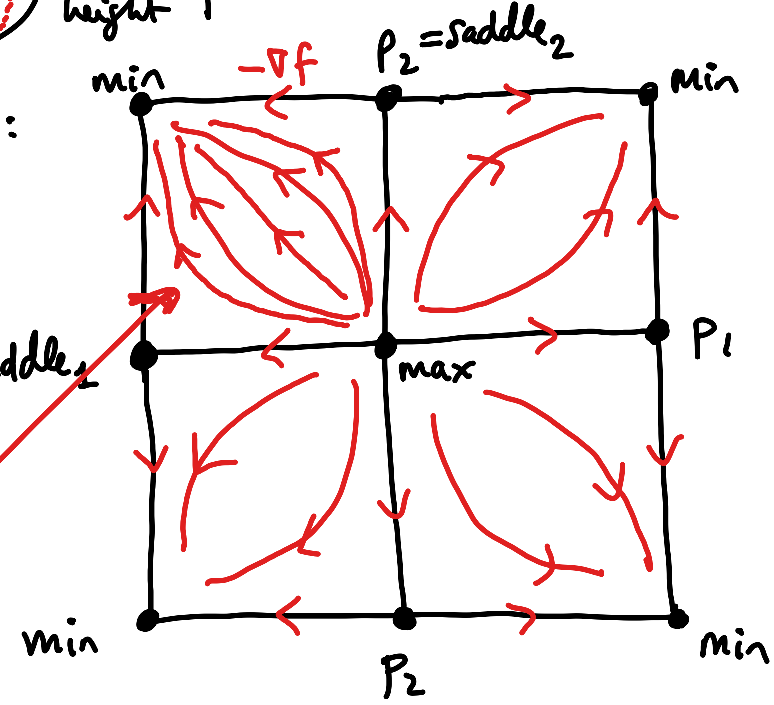
(torus slightly tilted)

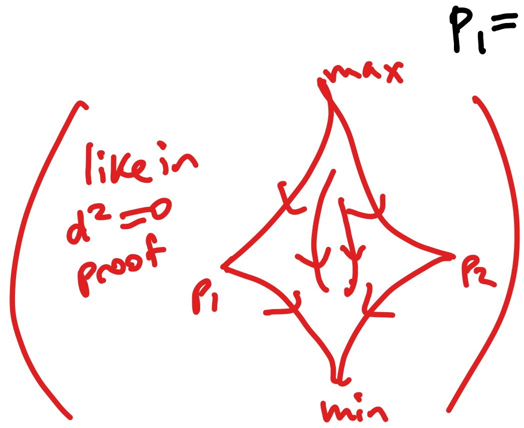
easy model:

recall $T^2 =$



P_1 = saddle$_1$

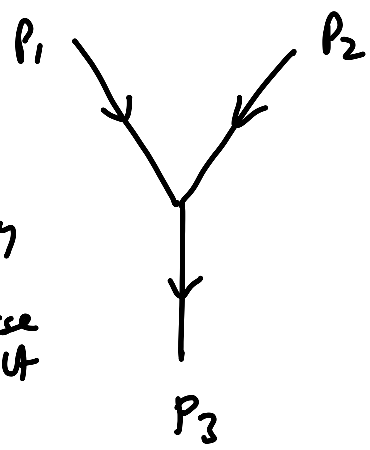




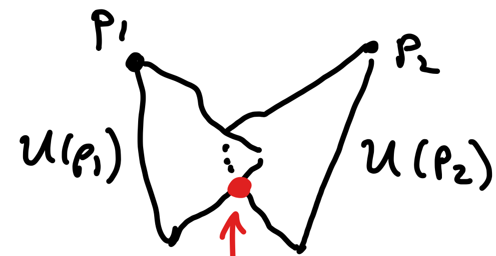
Try to also check cup product

... Why cup product?

homology version of Morse product



$\rightsquigarrow$ via $MC_*(f) \simeq C_*(M)$



pseudocycles representing p_1, p_2

looking at intersection points

generically,

$\mathcal{U}(p_1) \cap \mathcal{U}(p_2)$ represents classical intersection product on cycles

recall this is Poincaré dual to cup product on cohomology

how do we go back to $MH_*(f)$?

roughly: $C_*(M) \longrightarrow MC_*(f)$

$$\text{cycle } C \longmapsto \sum \#(C \cap \underbrace{\mathcal{D}(p_3)}) \cdot p_3$$

stable mfd (flowlines flowing into p_3)

Overview of Floer homology

Symplectic manifold

M smooth mfd

ω nondegenerate closed 2 form

$d\omega = 0$

locally $\omega = \sum \underbrace{A_{ij}}\, dx^i \wedge dx^j$

antisymmetric matrix

nondeg. means this matrix is nonsingular

forces $\dim M = 2n$ even

nondeg. equivalent to: ω^n is a volume form for M.

example $\mathbb{R}^2$ $\quad x^1 = x, \quad x^2 = y$ $\qquad (2n=2)$

$\mathbb{C}$, $z = x+iy$

$\omega = dx \wedge dy$ area form

matrix $A = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

examples
- $\mathbb{C}^n$, $\quad \omega = \sum dx_j \wedge dy_j \qquad z_j = x_j + iy_j$
- Σ = orientable surface $\qquad \omega$ = area form
- T^*N ← cotangent bundle, in local trivialisn (q_j, p_j) $\qquad \omega = \sum dp_j \wedge dq_j$

  any mfd N; coords on N; for fiber
- Kähler manifolds

  (M, I) cx mfd $\qquad$ Kähler metric h

  $\Rightarrow$ Riem. metric $g = \mathrm{Re}(h)$

  Sympl. form $\omega = \mathrm{Im}(h)$

  compatible $\boxed{g = \omega(\cdot, I\cdot)}$ $\quad$ (since $I^2 = -\mathrm{id}$: $\omega = -g(\cdot, I\cdot)$)

I complex structure

$I : TM \to TM$ linear, $I^2 = -\mathrm{id}$

in local cx coords arises from mult^n by $i = \sqrt{-1}$

Darboux's Lemma $\exists$ local coords on M s.t.

$$\omega = \sum dx_j \wedge dy_j$$ (like $\mathbb{C}^n$ example)

$\Rightarrow$ sympl mflds don't have interesting local invariants like in Riem. geometry (curvature, ...)

$\Rightarrow$ want global symplectic invariants (motivation for Floer theory)

Why do we want ω closed?

$H : M \to \mathbb{R}$ smooth function "Hamiltonian"

$\Longrightarrow$ Hamiltonian vector field X_H:

$$\omega(\cdot, X_H) = dH$$

$\varphi^t :=$ flow for time t of X_H, $\varphi^0 = id$

claim ω closed $\Leftrightarrow$ flow preserves ω $\forall H$

$$(\varphi^t)^* \omega = \omega$$

pf

$$\underbrace{\frac{\partial}{\partial t}\Big|_t (\varphi^t)^* \omega}_{0 \text{ iff flow preserves } \omega} = (\varphi^t)^* \frac{\partial}{\partial s}\Big|_{s=0} \varphi_s^* \omega$$

$\varphi^{t+s} = \varphi^t \circ \varphi^s$

definition of Lie derivative

$$= (\varphi^t)^* \mathcal{L}_{X_H} \omega$$

Cartan's formula

$$= (\varphi^t)^* \Big(d \underbrace{i_{X_H} \omega}_{= \omega(X_H, \cdot) = -dH} + i_{X_H} d\omega \Big)$$

($i_V \alpha := \alpha(V, \ldots)$) V: v.f., α: form

$$= \underbrace{(\varphi^t)^*}_{\text{iso}} \underbrace{i_{X_H}}_{\text{any } H} d\omega = 0$$

iff ω closed (if allow varying H)

since $ddH = 0$ ($d^2 = 0$) ∎

$\leftarrow$ any H $\leftarrow$ locally can make X_H(chosen point) = any vector

compare: Riem. case

$$g(\cdot, \nabla H) = dH$$

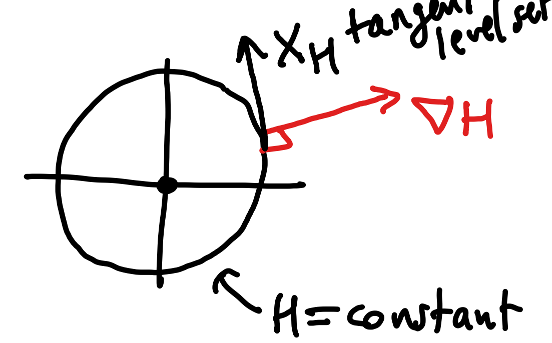
in Kähler case $g = \omega(\cdot, I \cdot)$

exercise:

$$X_H = I \nabla H$$

think "rotate $90°$ the gradient"

$\mathbb{C}$: $H = |z|^2$



flow X_H is rotation about 0

<u>exercise</u> in Darboux chart (i.e. Darboux lemma coords)

find explicit equations for

$$\frac{dx}{dt} = X_H$$

should get the classical "<u>Hamilton equations</u>" from classical mechanics.

called "<u>canonical coords</u>" in classical mechanics

A <u>symplectomorphism</u>

$$(M_1, \omega_1) \xrightarrow{\psi} (M_2, \omega_2)$$

means

- ψ diffeomorphism
- $\psi^* \omega_2 = \omega_1$

in classical mechanics if locally use Darboux coords, symplectomorph means "<u>canonical transformations</u>" in classical mechanics