# Morse Homology 3

**Last time:** closed sympl. mfd $(M,\omega)$

$H: M \to \mathbb{R}$ $\qquad \omega(\cdot, X_H) = dH \leftarrow$ Hamiltonian vector field $X_H$

$\omega$ closed $\Rightarrow$ flow $\varphi^t$ of $X_H$ preserves $\omega$

$\varphi: (M_1,\omega_1) \to (M_2,\omega_2)$ symplectomorphism means $\left\langle \begin{array}{l} \text{diffeomorphism} \\ \varphi^*\omega_2 = \omega_1 \end{array} \right.$

**Example**

$\varphi = \varphi^1 : M \to M$ time 1 flow of $X_{H_t}$ $\leftarrow$ allow time dependence $H: M \times S^1 \to M$, $H_t = H(\cdot, t)$, $\omega(\cdot, X_{H_t}) = dH_t$

- symplectomorphism
- $\mathrm{Fix}(\varphi) = \{1\text{-periodic orbits of } X_{H_t}\}$

  $x_0 \longleftrightarrow (x(t) = \varphi^t(x_0))$

If time indep. $H = H_t$ then $\mathrm{Crit}(H) = \{\text{constant 1-periodic orbits}\} \subseteq \mathrm{Fix}(\varphi)$

**Arnol'd Conjecture**

$$\#(1\text{-periodic orbits of } X_{H_t}) \geq \sum \dim H_i(M;\mathbb{R})$$

$\uparrow$ Betti numbers

**Floer:**
- build homology involving $\mathrm{Fix}(\varphi)$ as chain level generators "$HF^*(H_t)$"
- prove invariant independent of choice of Hamiltonian $H_t$ ⊛

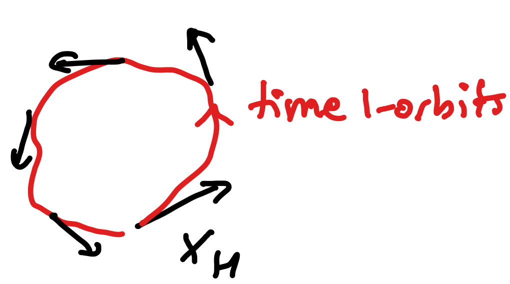
$$\begin{aligned} \#\mathrm{Fix}(\varphi) &= \dim(\text{chain complex}) \\ &\geq \dim HF_*(H_t) \\ &= \dim HF_*(C^2\text{-small Morse time indep. } H) \\ &= \dim MH_*(H) \\ &= \dim H_*(M;\mathbb{R}) \quad \checkmark \end{aligned}$$

$H: M \to \mathbb{R}$ Morse function

invariance ⊛

last time: Morse homology $\cong$ singular homology.

idea: Why $C^2$-small?




$\left\{ \begin{array}{l} H \text{ small} \\ 1^{st} \text{ derivs small} \end{array} \right.$

if $H$ is $C^1$-small

$\Rightarrow X_H$ small (using choice Riem. metric $g$)

$\Rightarrow$ cannot flow far in time

$\Rightarrow$ 1-orbit stays in small chart

if $C^2$-small, so $2^{nd}$ derivs small $\Rightarrow X_H$ cannot rotate fast enough to give rise to closed 1-orbit unless constant orbit

$\Rightarrow \mathrm{Fix}(\varphi) = \mathrm{Crit}(H)$ constant orbits.

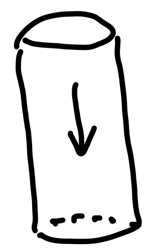
$\Rightarrow$ chain level generators agree with Morse cx

<u>Fact</u> (under suitable assumptions on $M$) Floer differential becomes Morse differential

$$HF^*(C^2 \text{ small Morse } H) \cong MH_*(H)$$

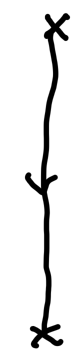
explain below, picture:

1-periodic orbits of $H$



cylinders $\mathbb{R}\times S^1 \to M$ satisfying a certain PDE called Floer's equation

$H$ $C^2$ small Morse $\rightsquigarrow$



Crit

time indep. cylinders so $\mathbb{R}\to M$ and $\partial_s u = -\nabla H$

<u>Floer</u> define "Morse theory" on space of free loops

free loop space $\quad \mathcal{L}M = C^\infty(S^1, M)$

smooth maps $x: S^1 \to M$
<u>no</u> choice of base point

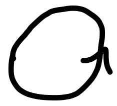
define function
$$A_H : \mathcal{L}M \to \mathbb{R}$$
so that

$$\begin{cases} \text{Crit } A_H = \text{1-periodic orbits of } X_H \\ -\nabla A_H \text{ trajectories} = \text{PDE solutions above} \end{cases}$$

<u>Symplectic action functional</u>

$$A_H(x) = -\int \underbrace{\bar{x}^*\omega} + \int_0^1 H(x(t))\,dt$$

$x: S^1 \to M$ $\qquad$ 2 form on disc $\mathbb{D}$ $\qquad$ $S^1 = \mathbb{R}/\mathbb{Z}$



$x$

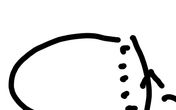
assume $x$ contractible
so $A_H : \mathcal{L}_0 M \to M$

contractible loops

is this well-defined indep. of choice of "filling" $\mathbb{D} \xrightarrow{\bar{x}} M$

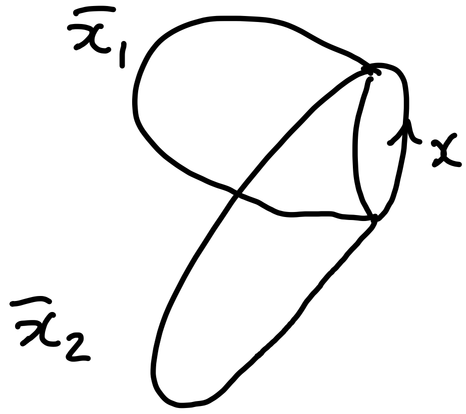
<u>No!</u>

$\bar{x}$



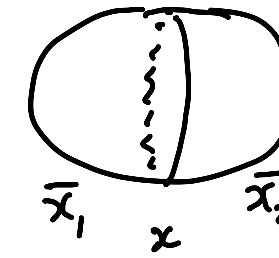
$x$

choice of contraction to point
so smooth disc $\bar{x} : \mathbb{D} \to M$




glue these $\bar{x}_1 \# (-\bar{x}_2)$

reverse orientation




$\Rightarrow$ sphere $v : S^2 \to M$

$$A_H(\bar{x}_1) - A_H(\bar{x}_2) = \int_{S^2} v^* \omega$$

One fix: old papers on Floer theory see the assumption "$\omega$ is aspherical"

means

$[\omega](A) = 0$ for sphere classes $A \in \mathrm{image}(\pi_2 M \to H_2 M)$

Hurewicz

$\left(\Rightarrow \int_{S^2} v^* \omega = 0\right)$ (pairing $H^2(M) \times H_2(M) \to \mathbb{R}$)

Another fix: $A_H : \widetilde{\mathcal{L}_0 M} \longrightarrow \mathbb{R}$

Better

cover of free loop space consisting of pairs $(x, \bar{x})$

smooth $\bar{x} : \mathbb{D} \to M$

$\bar{x}|_{S^1 = \partial \mathbb{D}} = x$

<u>Rmk</u> $M$ non-compact sympl. mfd then $\exists$ large class of examples for which $\omega$ is exact

$$\omega = d\theta$$

example: $M = T^* N$

$$\omega \overset{\text{locally}}{=} \sum d\underset{\text{fiber coords}}{p_j} \wedge d\underset{\text{base coords}}{q_j}$$

$$= d\Big(\underbrace{\sum p_j \, dq_j}_{\text{"canonical 1-form" } \theta}\Big)$$

essentially $\theta_{(q,p)} = p$

- $\omega$ aspherical:

$$\int_{S^2} v^* \omega = \int_{S^2} v^* d\theta$$
$$= \int_{S^2} d(v^* \theta)$$
$$\text{Stokes's Theorem} \longrightarrow = \int_{\partial(S^2) = \emptyset} v^* \theta = 0$$

$\bar{x} : \mathbb{D} \to M$

in case $\omega = d\theta$

$$A_H(x) = \underbrace{-\int_{\mathbb{D}} \bar{x}^* d\theta}_{\begin{gathered} \| \\ -\int_{\mathbb{D}} d(\bar{x}^*\theta) \\ \| \\ -\int_{\partial(\mathbb{D}) = S^1} \bar{x}^*\theta \\ \| \\ -\int_{S^1} x^*\theta \end{gathered}} + \int_{S^1} H(x)\, dt$$

$$\Longrightarrow \quad A_H(x) = -\int_{S^1} x^*\theta + \int_{S^1} H(x)\, dt$$

for $T^*N$:
$\theta =$ "$p\,dq$"

$$A_H(x) = -\int_x p\, dq + \int_x H$$

classical action functional from mechanics

<u>Rmk</u> $M^n$ closed sympl mfd then $\omega$ <u>cannot</u> be exact:

$\omega = d\theta$

$\omega^n$ is volume form (by non-degeneracy of $\omega$) $\leftarrow$ general $\forall$ sympl mfd

$$0 < \int_M \omega^n = \int_M d\theta \wedge \dots \wedge d\theta = \int_M d(\theta \underbrace{d\theta \wedge \dots \wedge d\theta}_{n-1 \text{ copies}})$$

$$\overset{=}{\text{Stokes}} \int_{\underset{\underset{M \text{ closed mfd}}{\uparrow}}{\partial M = \emptyset}} \theta\, d\theta \wedge \dots \wedge d\theta$$

$= 0$ $\lightning$

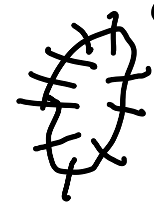
<u>Example</u> $T^2 = S^1 \times S^1$ with area form $\omega$ is symplectic
no interesting spheres : $\pi_2(T^2) = 0$ so aspherical
but closed mfd so not exact.

**Claim** Crit $A_H$ = {1-periodic orbits of $X_H$}

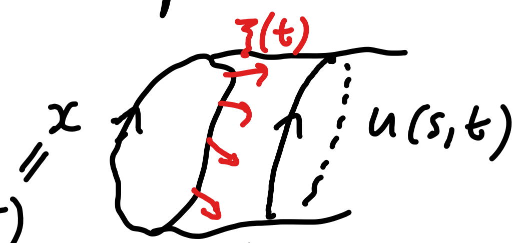
**proof**

$$T_x \boxed{\mathcal{L}_0 M} \ni \xi$$

could define mfd structure in detail
charts using exp map etc.



simpler point of view:




$$u(0,t) = x(t)$$

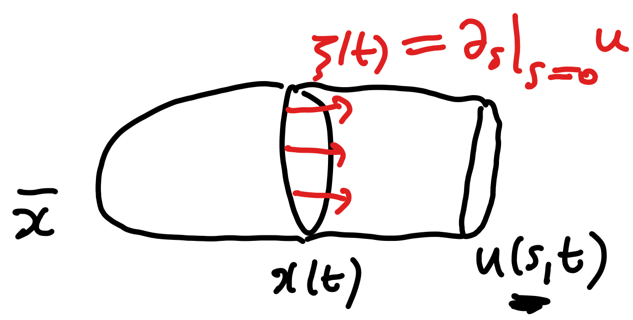
deform $x$ in $\mathcal{L}_0 M$

$$\xi(t) = \partial_s\big|_{s=0} u(s,t)$$

$=$ smooth vector field parametrized by $t \in S^1$

$\xi(t) \in T_{x(t)} M$

$$T_x \mathcal{L}_0 M = \Gamma(x^* TM) \leftarrow \text{"smooth vector fields along } x\text{"}$$




$u(s,t)$ extends $\bar{x}$

$$u(s,t) = \bar{x}\left(e^{2\pi(s+it)}\right)$$

$$(s,t) \in (-\infty, 0] \times S^1$$

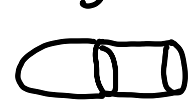
$$s = -\infty \leftrightarrow 0 \in D$$

$$0 = d_x A_H \cdot \xi = \frac{\partial}{\partial s}\Big|_{s=0} A_H(u(s,t))$$

assume $x$ critical pt of $A_H$

by definition essentially

$$= \frac{\partial}{\partial s}\Big|_{s=0} \left( -\int u^* \omega + \int_{S^1} H(u_s)\, dt \right)$$



loop $u_s = u(s,\cdot)$

$$= \partial_s\Big|_{s=0}\Bigg( - \overbrace{\int_{x=u_0,\ u_s} u^*\omega}^{= \iint \omega(\partial_s u, \partial_t u)\, ds\, dt} + \int H(u_s)\,dt \Bigg)$$

$$-\int \left[\lim_{\delta\to 0} \frac{\displaystyle\int_{u_0,\ u_\delta} u^*\omega - \overbrace{\int_{u_0} u^*\omega}^{=0}}{\delta}\right] dt$$

$$\int \partial_s\Big|_{s=0} H(u_s)\,dt = \int dH\big(\partial_s|_{s=0} u_s\big)\,dt = \int dH\big(\xi(t)\big)\,dt = \int \omega\big(\xi(t), X_H\big)\,dt$$

$$\omega(\underbrace{\partial_s u}_{\approx\, \xi(t)}, \underbrace{\partial_t u}_{\approx\, \partial_t x})\, ds\, dt$$

$\approx \partial_t x$ for $\delta$ small since $u(0,t) = x(t)$

$\approx \xi(t)$ when $s \in [0,\delta]$ and $\delta$ small

continuous (smooth!) in $s$

$$-\int \lim_{\delta\to 0}\left( \frac{\displaystyle\int_{0,\ \delta} \omega(\xi(t), \partial_t x)\cdot ds\, dt}{\delta} + \text{small error}\right)$$

$$-\int \lim_{\delta\to 0}\left( \frac{\cancel{\delta}\cdot \omega(\xi(t), \partial_t x)\, dt}{\cancel{\delta}} + \text{small error}\right)$$

$$-\int_{S^1} \omega(\xi(t), \partial_t x)\, dt$$

$$\Rightarrow 0 = \partial_s A_H \cdot \xi = -\int_{S^1} \omega(\xi, \partial_t x)\, dt + \int_{S^1} \omega(\xi, X_H)\, dt$$

$$= -\int_{S^1} \omega(\xi, \partial_t x - X_H)\, dt$$

$\forall \xi \overset{\omega \text{ nondeg.}}{\Longrightarrow} \partial_t x - X_H = 0$ so $x$ is 1-orbit of $X_H$. ∎

# Floer's equation?

Preliminary remark about <u>almost complex structures</u>

<u>Def</u> a.c.s. $J$ means smooth automorphism

$$J : TM \to TM$$

with

$$J^2 = -id$$

| e.g. $\mathbb{C}$ |
|---|
| $\omega = dx \wedge dy$ |
| $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =$ rotat$^n$ by $90°$ |
| $\omega(\cdot, J\cdot) = dx^{\otimes 2} + dy^{\otimes 2}$ $=$ usual $g$ |

<u>Example</u> Complex mfd $J$ induced by multiplication by $i = \sqrt{-1}$ in local coordinates

Why "almost"? Don't assume $J$ arises from local complex coordinates and $i = \sqrt{-1}$.

may not exist

even though $T_pM$ is a complex vector space by defining $i \cdot v = J(v)$, $v \in T_pM$

RmK $\Rightarrow TM$ is cx vector bundle so can talk about Chern classes

<u>Def</u> $J$ is <u>$\omega$-compatible</u> if

$$g = \omega(\cdot, J\cdot) \quad \text{any Riem. metric}$$

e.g. Kähler mfd $g$ (in particular $\omega(J\cdot, J\cdot) = \omega(\cdot,\cdot)$)

<u>Fact</u> $\forall$ sympl. mfd $(M, \omega)$ the space

$$\{\omega\text{-compatible a.c.s. } J\}$$

is a non-empty contractible space.

$\exists J$

if show that Floer homology is invariant up to isomorphism under deforming $J$ then contractibility $\Rightarrow$ choice of $J$ does not matter up to iso

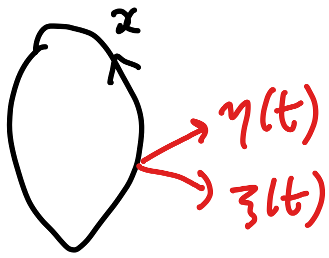
"Auxiliary data"
want your homology indep. of it.

$\Longrightarrow$ $L^2$-Metric on $\mathcal{L}M$

have chosen $\omega$-compat. a.c.s.

$$T_x\mathcal{L}M \times T_x\mathcal{L}M \longrightarrow \mathbb{R}$$

$$(\xi, \eta) \longmapsto \int_{S^1} \underbrace{\omega(\xi(t), J\eta(t))}_{= g(\xi(t), \eta(t))} \, dt$$




claim The $-\nabla A_H$-flowlines satisfy Floer's equation

$$u : (\text{subset of } \underset{(s,t)}{\mathbb{R} \times S^1}) \longrightarrow M$$

$$\partial_s u + J(\partial_t u - X_H) = 0$$

proof $\partial_s u = -\nabla A_H$

definition $\langle \cdot, \nabla A_H \rangle = dA_H$

$$\langle \xi, \nabla A_H \rangle = dA_H \cdot \xi$$

$$\underset{\text{before}}{=} -\int_{S^1} \omega(\xi, \partial_t x - X_H) \, dt$$

$J^2 = -id$

$$= \int_{S^1} \omega(\xi, J^2(\partial_t x - X_H)) \, dt$$

$$= \int_{S^1} \omega(\xi, J \underbrace{(J(\partial_t x - X_H))}) \, dt$$

$$= \int_{S^1} g(\xi, J(\partial_t x - X_H)) \, dt$$

$\forall \xi \Longrightarrow$

$$\nabla A_H = J(\partial_t x - X_H) \qquad \blacksquare$$

Rmk Wrote $H$ everywhere but $H_t$ also OK, does not affect calculations.

$$\mathcal{L}M = C^\infty(S^1, M)$$

## Moduli space of Floer trajectories

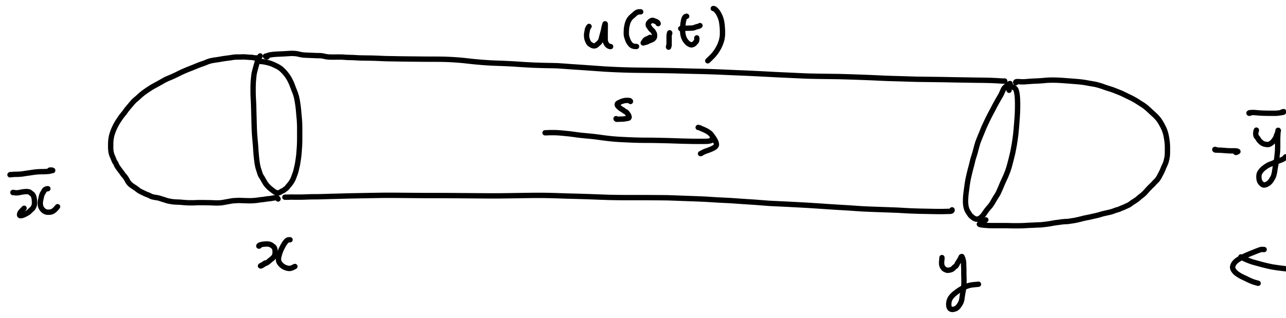
$$\mathcal{M}(x,y) = \{u : \mathbb{R} \to \mathcal{L}M, \quad \partial_s u = -\nabla A_H\} / \mathbb{R}\text{-repar.}$$

$$= \{u : \mathbb{R}\times S^1 \to M, \quad \partial_s u + J(\partial_t u - X_H) = 0\}$$

$$u \sim u(\cdot + \text{constant}, \cdot)$$

$\uparrow$ $\mathbb{R}$ coord

More precisely, if we keep track of fillings so work with $\widetilde{A_H} : \mathcal{L}_0 M \to \mathbb{R}$




$\leftarrow$ get "sphere" $S^2 \to M$

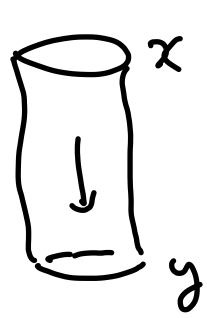
$$\mathcal{M}(\bar{x}, \bar{y}) = \{\text{above } u : [\bar{x} \# u \# (-\bar{y})] = 0 \in H_2(M)\}$$

## Floer chain complex

$$CF^*(H) = \bigoplus_{x \in \mathrm{Crit}(A_H)} \mathbb{Z}/2 \cdot \bar{x}$$

$$\partial \bar{x} = \sum_{\dim \mathcal{M}(\bar{x},\bar{y}) = 0} \#(\mathcal{M}(\bar{x},\bar{y})) \cdot \bar{y}$$

idea




$x$ 1-periodic orbits of $H$

$\leftarrow$ Floer cylinders $u : \mathbb{R}\times S^1 \to M$ satisfying Floer's PDE eqn

$$\partial_s u + J(\partial_t u - X_H) = 0$$

# Many difficulties in details

1) H time indep $\Rightarrow$ nonconstant 1-orbits are never isolated

$$\partial_t x = X_H \Rightarrow \text{also } x(\cdot + \text{const}) \text{ is 1-orbit}$$

$\therefore$ have to make time-dependent perturbation of $H$ to make 1-orbits "non-degenerate"

analogue of Morse crit. pts. where ask nondeg. of Hessian

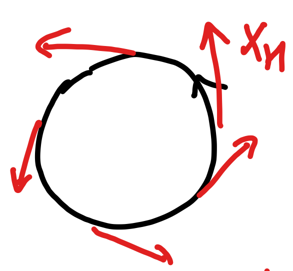
Here:

nondeg of Hess $A_H$ at crit pt

$\leftrightarrow$ differential $d\varphi^1_{H_t}$ of time 1 flow map has

no eigenvalue $= 1$

i.e. $\ker(d\varphi^1_{H_t} - \mathrm{id}) = \{0\}$





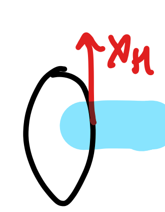
evector $d\varphi^1 \cdot X_H = 1 \cdot X_H$
if $H$ time indep.

## Alternative: "Morse-Bott theory"

one allows critical sets that are submfds

Fact generic time indep. perturbation of time-indep. $H$ gets rid of all 1-evectors except obvious 1-evecs $\mathbb{R} \cdot X_H$

"transversally non-degenerate"





Morse-Bott function $f: M \to \mathbb{R}$ ← closed mfd

$\mathrm{Hess}_p f$ nondegenerate in directions normal to critical locus $\mathrm{Crit}(H)$

so allow obvious 0-eigenvalues caused by $T_p \mathrm{Crit}(H) \subseteq T_p M$

$$\mathrm{Hess} = \begin{pmatrix} \lambda \neq 0 & & \\ & 0 & \\ & & 0 \end{pmatrix} \leftarrow T(\text{max/low})$$

e.g. torus lying flat

$S^1 \cong$ max

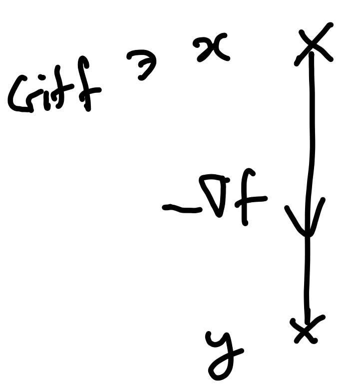
$S^1 \cong$ min

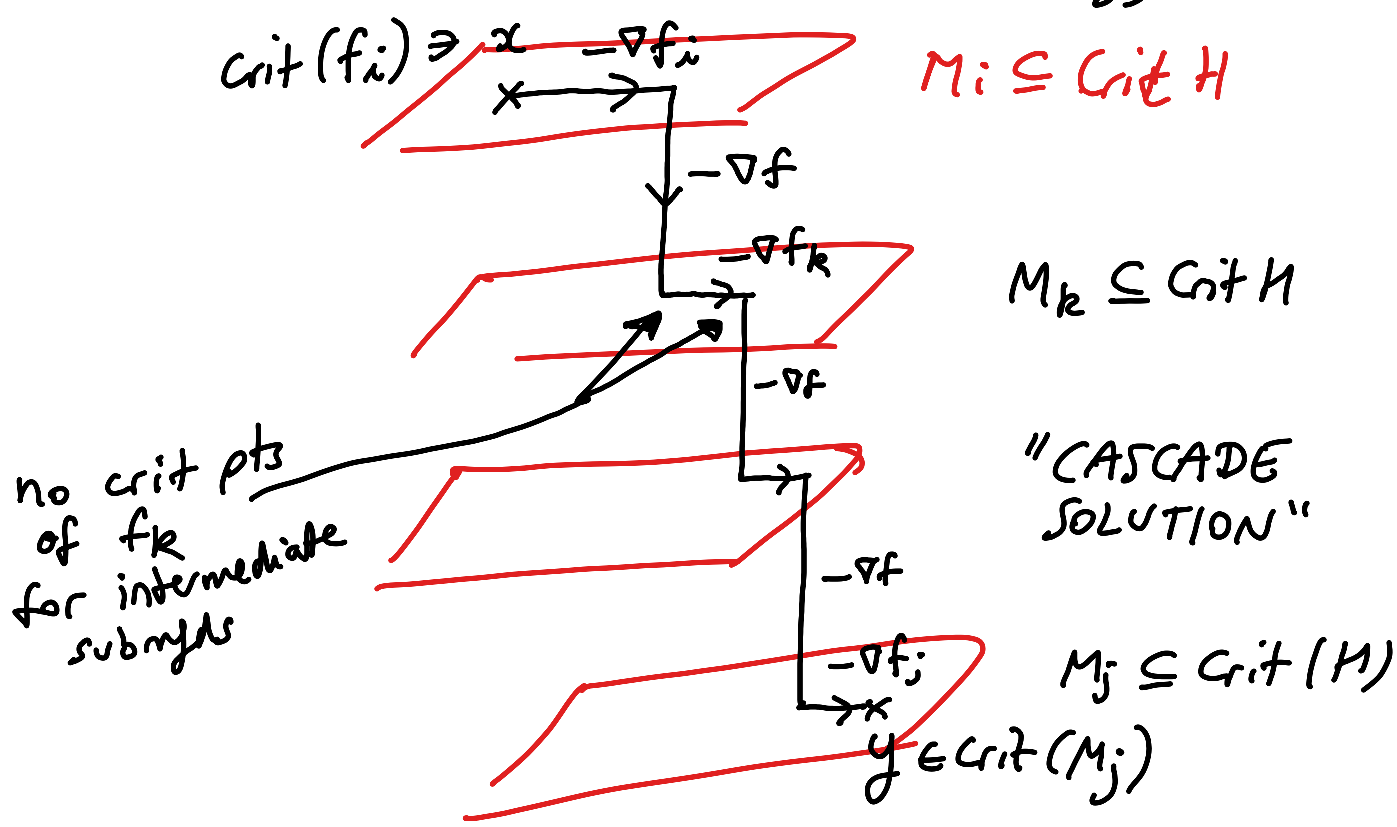
height $\rightarrow$ $\uparrow \mathbb{R}$

Morse–Bott homology: $\mathrm{Crit}(f) = \bigsqcup M_i$ ← submfds $M_i \subseteq \mathbb{R}$ ← Morse–Bott submfds

<u>Trick</u> choose auxiliary Morse functions $f_i : M_i \to \mathbb{R}$
then Morse–Bott chain differential counts isolated "CASCADES"

Morse homology vs Morse Bott homology




<u>Floer theory</u> for time-indep $H$ after perturb$^{n}$ to make it transversally nondeg

$$\mathrm{Crit}(\mathcal{A}_H) = \mathrm{Crit}(H) \sqcup \bigsqcup M_i$$

constant orbits

nonconstant orbits
$M_i \cong S^1$
choice of initial point

so can do
"Morse–Bott–Floer homology"
by choosing auxiliary
Morse functions $f_i : M_i \cong S^1 \to \mathbb{R}$
and count "CASCADES"