Last time Morse-Bott homology

$f: M \to \mathbb{R}$ Morse-Bott (Hess is nondeg in normal bundle to Crit f)

$\mathrm{Crit} f = \sqcup M_i$ ← submfds

$f_i : M_i \to \mathbb{R}$ auxiliary Morse fns

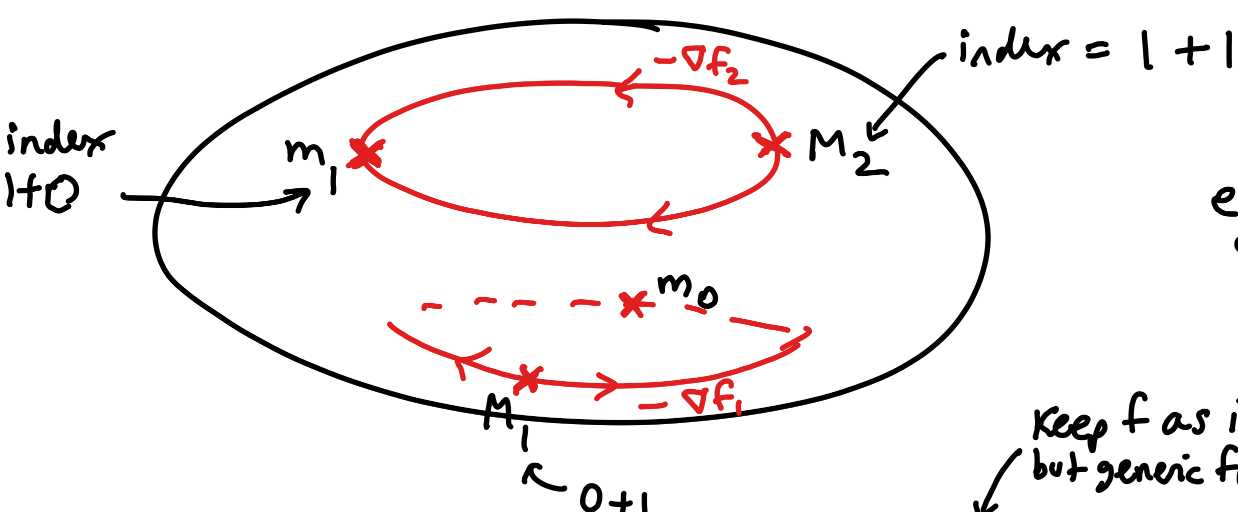
MBC_* generated freely by $\cup \mathrm{Crit} f_i$

∂ counts "cascades"

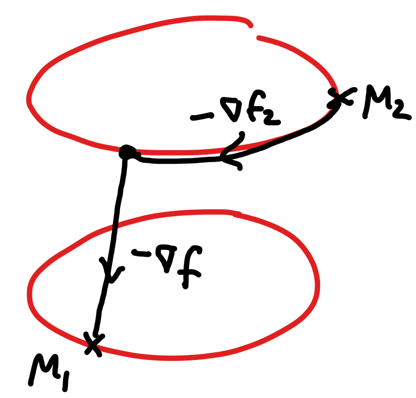
grading $|x| = \mathrm{MBindex}(x) + \mathrm{MorseIndex}_{f_i}(x)$

$\in \mathrm{Crit} f_i$ — # neg. evals of $\mathrm{Hess}_f(x)$ — # neg. evals of $\mathrm{Hess}_x(f_i)$

example



example of cascade

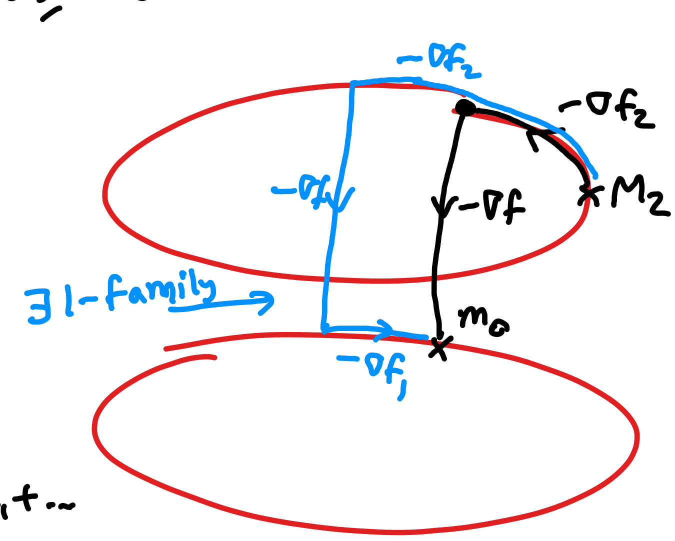


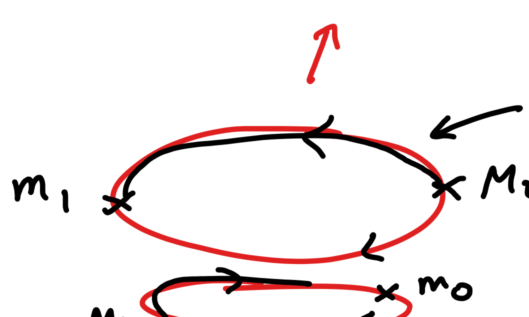
Keep f as is, but generic f_i, metric g

if everything is chosen generically, occurs if index difference = 1

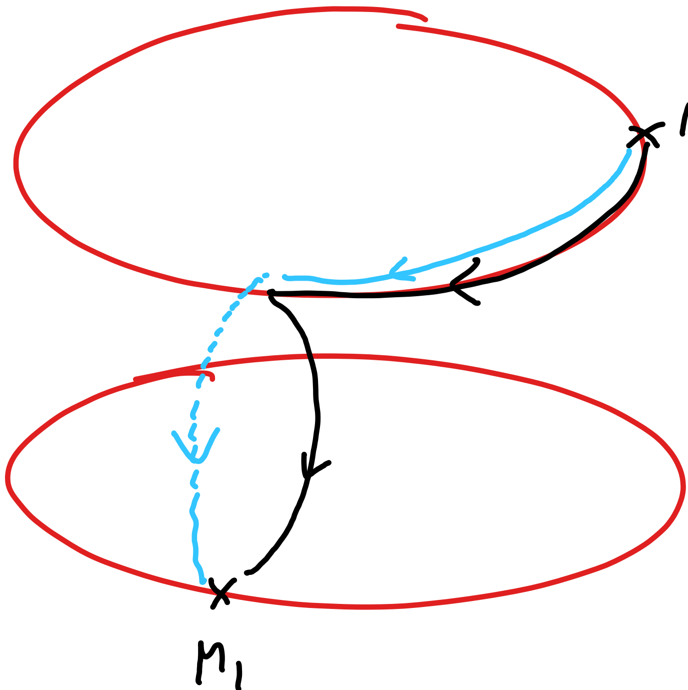
only count isolated cascades, so not:

(except in special case $x \xrightarrow{\nabla f_i} y$ where we quotient by $\mathbb{R}$-transl^n)





contributes to $\partial M_2 = \dots m_1 + \dots$



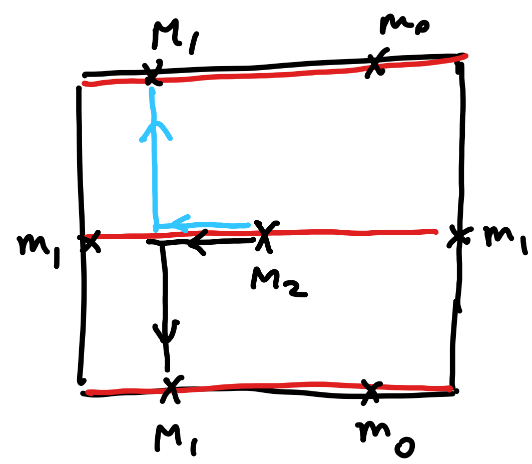
← also cancel

$\therefore \quad \partial M_2 = 0$

similarly get $\partial = 0$

$$\Rightarrow \quad MBH_*(T^2, f, f_i) \cong \begin{cases} \mathbb{Z}_2 m_0 & *=0 \\ \mathbb{Z}_2 m_1 \oplus \mathbb{Z}_2 M_1 & *=1 \\ \mathbb{Z}_2 M_2 & *=2 \end{cases}$$

↑ square picture: of T^2



<u>Floer case</u> $\exists$ time independent perturbation of $H: M \to \mathbb{R}$ (M,ω) so that the critical set of A_H is Morse-Bott for A_H

"transversally non-degenerate 1-periodic orbits"

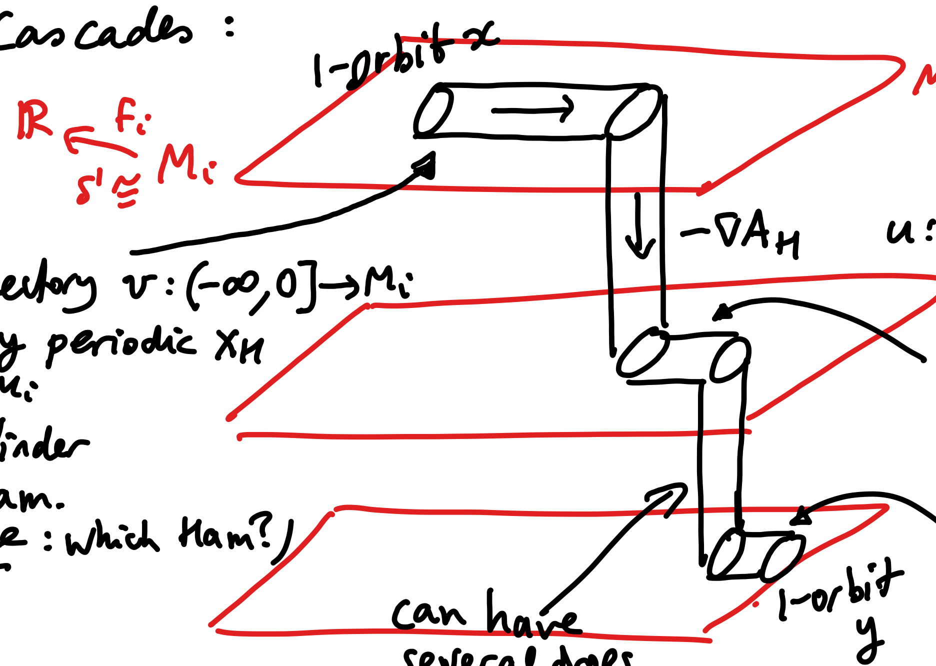
$$\operatorname{Ker}(d\varphi^1_H - id) = \mathbb{R} \cdot X_H$$

flow of X_H

Morse-Bott nbhds $\cong S^1$

choice of initial point $x(0)$ of 1-orbit

Cascades:



$u: \mathbb{R}\times S^1 \to M$ Floer cylinders for H

$-\nabla f_i$ trajectory $v: (-\infty, 0] \to M_i$ then apply periodic X_H flow on M_i

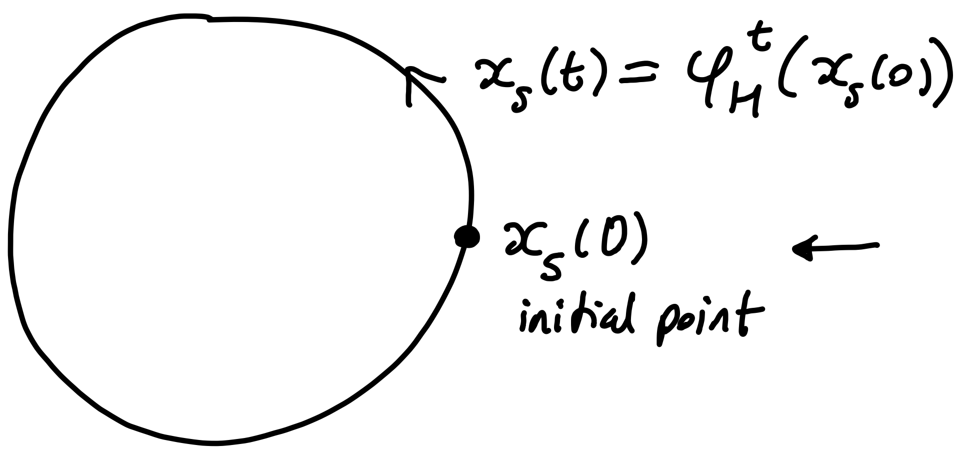
$\Rightarrow$ Floer cylinder some Ham.
(<u>exercise</u>: which Ham?)

similar but $v: [0, s_0] \to M_j$
finite $s_0 > 0$

$v: [0,\infty) \to M_k$

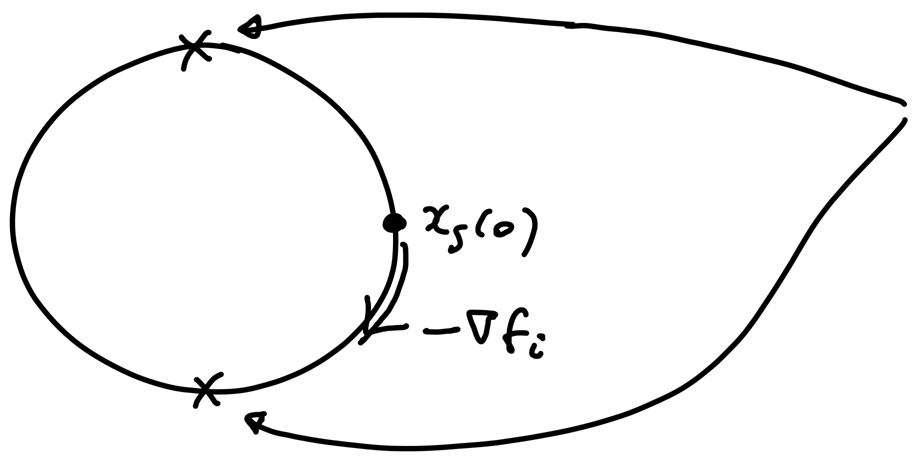
If no drops $\Rightarrow$ just doing Morse trajectories of f_i so $v: (-\infty,\infty) \to M_i$

what does cylinder in M_i look like?



$\longleftarrow \quad \partial_s(x_s(0)) = -\nabla f_i$

like reparametrising the 1-orbit as vary $s \in \mathbb{R}$



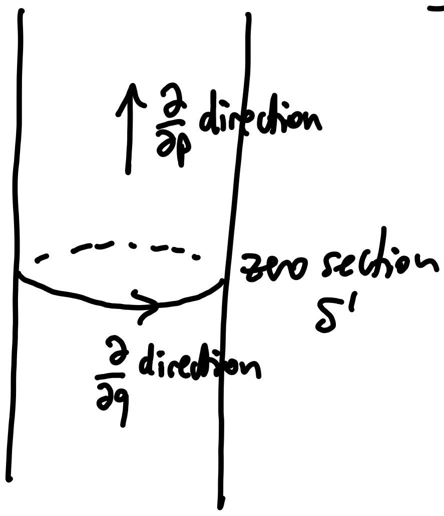
crit(f_i)
generators of MB Floer cx

Usually impossible to compute Floer cohomology because cannot solve PDE $\partial_s u + J(\partial_t u - X_H) = 0$

Lots of issues:
- flow of $-\nabla A_H$ is not defined for large time
- unstable and stable manifolds ($\mathcal{U}$ and $\mathcal{D}$) of A_H are typically ∞-dim'l

Very different from Morse case:
- $\partial_s u = -\nabla f$ ODE 1[st] order $\Rightarrow$ once know initial point $u(0)$ $\exists$ unique flowline $u(s)$ $\forall s \in \mathbb{R}$
- nice unstable/stable mfds: non-properly embedded copies of $\mathbb{R}^k$ ($k = \underset{\text{of crit pt}}{\text{index}}$ for $\mathcal{U}$, $\dim M -$ index for $\mathcal{D}$)

Example $M = T^*S^1$ recall $\omega = d(p\,dq) = dp\wedge dq$
$\cong S^1 \times \mathbb{R}$



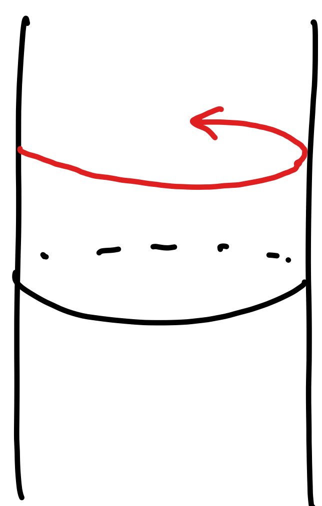
$H = \frac{1}{2}p^2$

$\omega(\cdot, X_H) = dH$
$= p\,dp$

$dp\wedge dq$

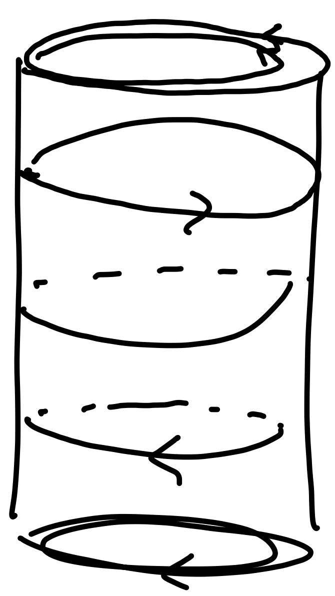
$\Rightarrow X_H = p\frac{\partial}{\partial q}$

(if identify $T^*S^1 \cong TS^1$
$\alpha \leftrightarrow v$
$\alpha = g(v,\cdot)$
fix some Riemannian g on S^1
$\Rightarrow$ this H is $\frac{1}{2}$ velocity^2
moment)



← flow of X_H becomes faster as p increases

← $p=0$ so $X_H=0$ so zero section ↔ constant 1-orbits



...
winding $w=2$ 1-orbits of X_H

winding # $w=1$ 1-orbits of X_H

$w=0$ constant orbits

$w=-1$

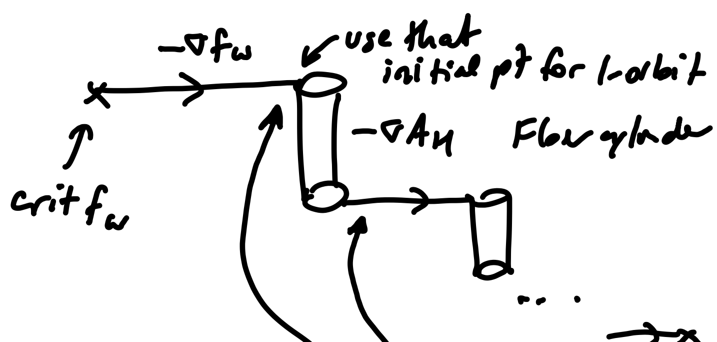
$w=-2$
...

$\Rightarrow$ $\boxed{\text{1-orbits of } X_H = \bigsqcup_{w\in\mathbb{Z}} S^1}$

Morse Bott Floer cx: $MBFC_* = \bigoplus_{w\in\mathbb{Z}} MC_*(f_w)$ ← auxiliary Morse $f_w : S^1 \to \mathbb{R}$

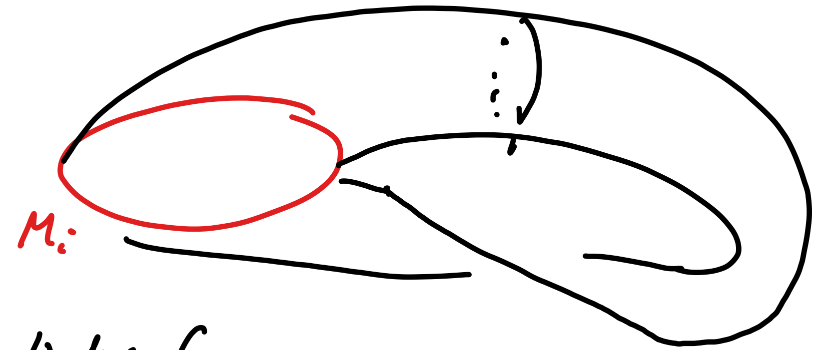
∂?

cascades:



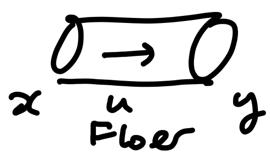
TRICK 1 if winding #s are different then $\nexists$ cylinder connecting the 1-orbits for topological reasons: winding # different

$\Rightarrow$ only worry whether



Floer cylinder from
1-orbit in M_i to another 1-orbit in <u>same</u> M_i

<u>claim</u> this is impossible, key idea will be an <u>a priori</u> <u>energy estimate</u>:



$$A_H(x) - A_H(y) = E(u) := \int_{\mathbb{R}\times S^1} |\partial_s u|^2 \, ds\, dt$$

If winding #s same $\Rightarrow$ A_H same at ends x, y $\Rightarrow$ $E(u) = 0$

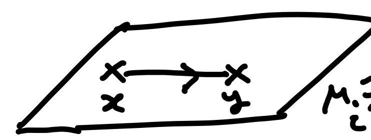
$\Rightarrow$ $\partial_s u = 0$ $\Rightarrow$ $\partial_t u - X_H = 0$

$\partial_s u + J(\partial_t u - X_H) = 0$

$\Rightarrow$ u is "constant" cylinder meaning s-independent 1-orbit of X_H.

<u>consequence of claim</u>

Only cascades are



$M_i \cong S^1$

$$\Rightarrow \quad MBFH_*(T^*S^2, H) \cong \bigoplus_{w\in\mathbb{Z}} MH_*(f_i)$$

$$\cong \bigoplus_{w\in\mathbb{Z}} H_*(M_i)$$

$$\cong \bigoplus_{w\in\mathbb{Z}} H_*(S^1)$$

$$\cong H_*(\mathcal{L}S^1)$$

free loopspace $S^1 \to S^1$

<u>Fact</u> (exercise) $\mathcal{L}S^1 \simeq \coprod_{w\in\mathbb{Z}} S^1$

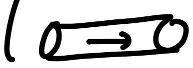
hpy equivalence

Instance of major theorem (Viterbo '96, also proofs by Abbondandolo-Schwarz, Salamon-Weber)

$$HF^*(T^*N, \text{quadratic Hamiltonian}) \cong H_{\dim N - *}(\mathcal{L}N)$$

closed mfd N

Various conflicting conventions in literature

FC^* I use: $\partial y = \sum \#\Big($ Floer  $\Big) \cdot x$

$\qquad\qquad\qquad\qquad\qquad x \qquad y$

cohomology conventions

"incoming cylinders"

(recall: Poincaré duality via Morse homology

if flip sign of H then like reversing s,t

$\tilde{u}(s,t) := u(-s,-t)$

$$\partial_s u + J(\partial_t u - X_H) = 0 \rightsquigarrow \partial_s \tilde{u} + J(\partial_t \tilde{u} - \underbrace{X_{-H}}_{=+X_H}) = 0$$

changes "incoming" to "outgoing" so get homological conventions

I use: $\omega(\cdot, X_H) = dH$

but some authors use

$i_{X_H}\omega = \omega(X_H, \cdot) = dH$ causing sign flip in H

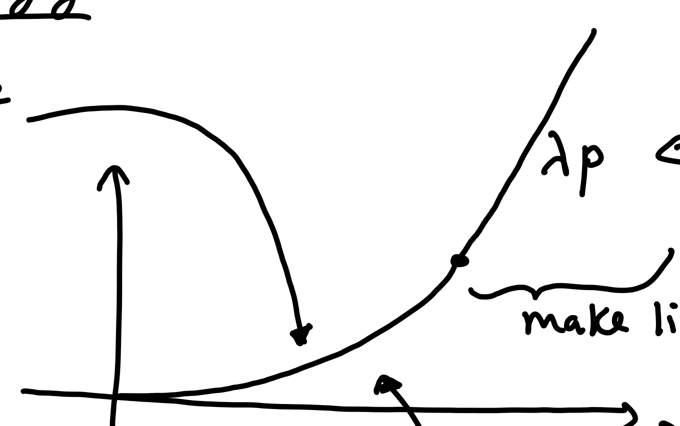
$\Rightarrow$ flips cohomology & homology

If do homological conventions + this convention then get again our cohomology conventions!

Symplectic homology

T^*S^1, $H = \frac{1}{2}p^2$

instead: $H_\lambda =$



here $X_H = \lambda \frac{\partial}{\partial q}$

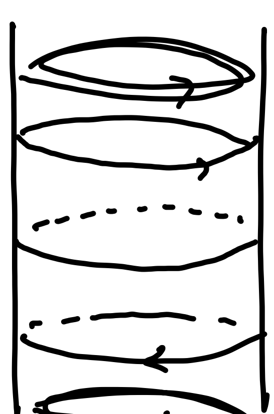
make linear at ∞

here $X_H = p\frac{\partial}{\partial q}$

pick λ generic (so that $\lambda\frac{\partial}{\partial q}$ on S^1 is not 1-periodic)

$q \in [0,1]$

$\lambda \notin \mathbb{Z}$



we only see 1-orbits with winding # $|w| \leq |\lambda|$

$$\Rightarrow MBFH_*(T^*S^1, H_\lambda) = \bigoplus_{-\lambda \leq w \leq \lambda} H_*(S^1)$$

$MBFH_*(T^*S^1, H) = H_*(\mathcal{L}S^1)$ $\xleftarrow[\lambda \to \infty]{\text{direct limit}}$

$\uparrow$

$MBFH_*(T^*S^1, \underset{\text{slope}>0}{H_{\text{small}}}) = H_*(\text{zero section } S^1)$

symplectic cohomology $\quad SH^*(T^*N) = \varinjlim_{\lambda\to\infty} HF^*(H_\lambda)$ "asymptotically linear Hams"

$$\begin{array}{ccc} SH^*(T^*N) & \cong & HF^*(\text{quadratic Ham } H) \\ \uparrow c^* & & \\ HF^*(H_{\text{small slope}>0}) & \cong H^*(T^*N) \cong & H^*(N) \end{array}$$

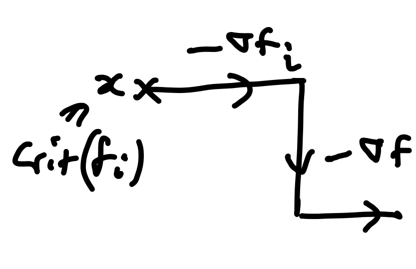
$\uparrow$ $T^*N \simeq N$ hpy equiv

Viterbo:

$$\begin{array}{ccc} SH^*(T^*N) & \cong & H_{\dim N - *}(\mathcal{L}N) \\ c^* \uparrow & & \uparrow \text{inclusion}_* \text{ of constant loops } S^1 \to N \\ H^*(T^*N) & & H_{\dim N - *}(N) \\ \text{hpy equiv } T^*N \simeq N \longrightarrow \parallel & \cong & \\ H^*(N) & & \end{array}$$

Poincaré duality for closed mfd N (work over $\mathbb{Z}/2$)

Poincaré duality via Morse-Bott (co)homology



$u_i : I_i \to M_i$

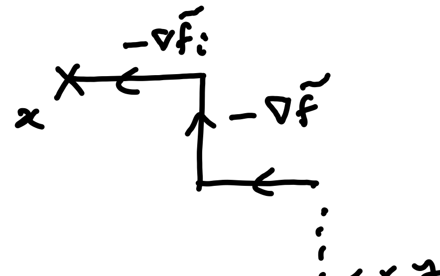
$v: \mathbb{R} \to M$

homological conventions

$\partial x = \sum \#(\text{outgoing cascades})\, y$

flip signs $\quad f_i \to -f_i = \tilde{f}_i$

$f \to -f = \tilde{f}$



now cohomological conventions

$\partial x = \sum \#(\text{incoming cascades}) \cdot y$

so same differential here

$\Rightarrow$ iso at chain level:

$$MBC_*(f, f_i) \longrightarrow MBC^{\dim M - *}(\tilde{f}, \tilde{f}_i)$$

$$x \longmapsto x$$

$\operatorname{Hess}_{\tilde{f}}(x) \cong \left(\begin{array}{c|c} + & 0 \\ \hline 0 & 0 \end{array}\right)$ $\#$zeros $= \dim M_i$ caused $\operatorname{crit}(f)$

new index$(x) = MB_{\tilde{f}}(x) + M_{\tilde{f}_i}(x) = (\dim M - \dim M_i - MB_f(x)) + (\dim M_i - M_{f_i}(x))$

$= \dim M -$ old index(x)

Rmk Just like for Morse homology, where invariance:

$$MH^*(M, f_1, g_1) \cong MH^*(M, f_2, g_2)$$

f_i: Morse $M \to \mathbb{R}$
g_i: generic Riem metrics

here also have invariance:

$$MBH^*(M, f_1, \overset{\text{auxiliary Morse}}{\mathrm{Crit}(f_1) \xrightarrow[h_1]{} \mathbb{R}}, g_1) \cong MBH^*(M, f_2, \mathrm{Crit}(f_2) \xrightarrow[h_2]{} \mathbb{R}, g_2)$$

f_i MorseBott $M \to \mathbb{R}$
g_i generic Riem metrics
h_i choice of generic auxiliary Morse functions

Cor $MBH^* \cong MH^*(f) \cong H^*(M)$

pf $MBH^* \underset{\text{invce}}{\cong} MBH^*(\text{Morse function } \underset{\text{crit } f = \sqcup \text{ points}}{f}) \overset{\text{exercise}}{\cong} MH^*(f)$

or:

$MBH^* \cong MBH^*(\text{MorseBott function } f=0) \overset{\text{exercise}}{\cong} MH^*\left(\begin{array}{c}\text{auxiliary Morse function}\\ \mathrm{crit}\, f = M \to \mathbb{R}\end{array}\right)$ ∎

Energy in Morse Theory

$f : M \to \mathbb{R}$ Morse
$u : \mathbb{R} \to M$ smooth
g Riem metric $\qquad g(\nabla f, \cdot) = df$

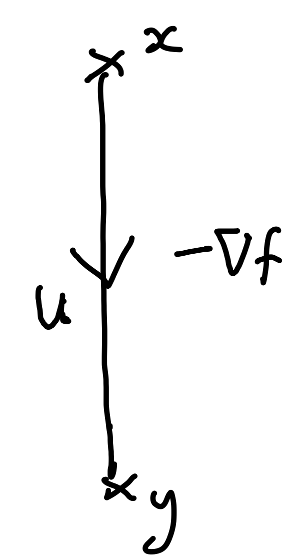
<u>energy</u> $\quad E(u) = \int_{\mathbb{R}} \underbrace{|\partial_s u|^2}_{= \, g(\partial_s u, \partial_s u) \geq 0} \, ds$

unlike length $L(u) = \int_{\mathbb{R}} |\partial_s u| \, ds$

$$\Rightarrow E(u) \begin{cases} \geq 0 \\ = 0 \Leftrightarrow \partial_s u \equiv 0 \end{cases}$$

If Morse traj : $\partial_s u = -\nabla f$:

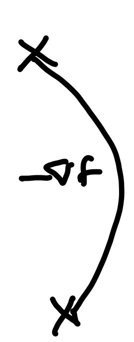
$$\begin{aligned} E(u) &= \int_{\mathbb{R}} g(-\nabla f, \partial_s u) \, ds \\ &= \int_{\mathbb{R}} -df(\partial_s u) \, ds \\ &= -\int_{\mathbb{R}} \partial_s (f \circ u) \, ds \\ &\underset{\text{Stokes/ fund. thm of calculus}}{=} f(x) - f(y) \end{aligned}$$

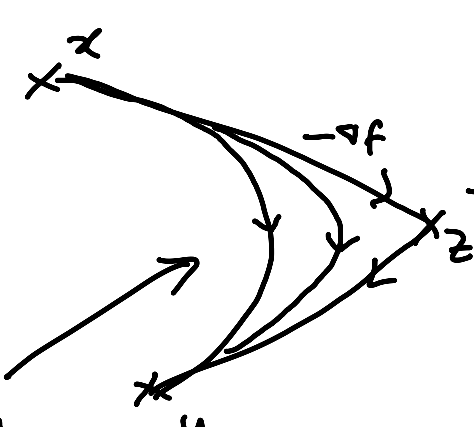


- Called "a priori energy estimate" since only depends on the "ends" x, y not on u.
- Obviously a homotopy invariant relative to the ends. so energy is constant for a family of Morse trajectories that have same ends x, y

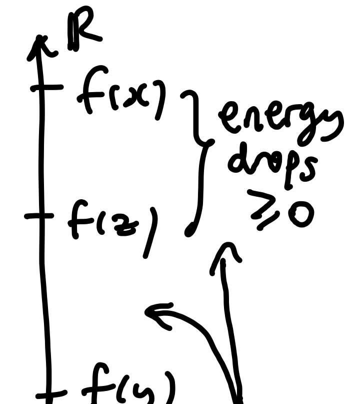
$\Rightarrow$ breaking

e.g. proof of $\partial^2 = 0$:





$\xrightarrow{f}$



1-family breaking

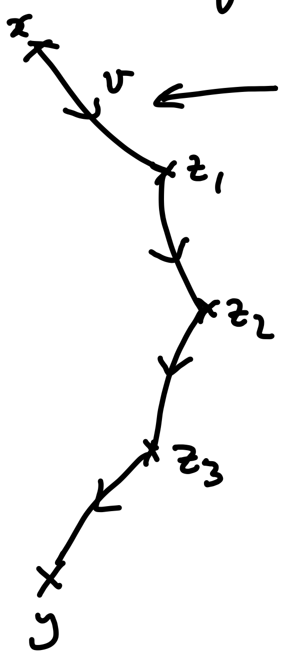
2 drops = total drop

$$(f(x) - f(z)) + (f(z) - f(y)) = f(x) - f(y)$$

E is preserved under breaking :

Claim Sequence u_n $-\nabla f$ trajectories that "converge" to broken flowline $\Rightarrow$ # breakings is finite

pf



$v \leftarrow$ non-constant Morse traj $\Rightarrow E > 0$

$\Rightarrow f(x) > f(z_1)$ not equality

TRICK energy is "quantized":

$$E(v) \geq \hbar := \min_{\substack{a,b \in \mathrm{Crit} f \\ f(a) \neq f(b)}} |f(a) - f(b)| > 0$$

M closed so compact so Crit f is finite

Morse! crit f are isolated pts

$$\Rightarrow \# \text{breakings} \leq \frac{\text{total } E}{\hbar} = \frac{f(x) - f(y)}{\hbar} \quad \blacksquare$$

Energy in Floer theory

$H : M \to \mathbb{R}$ (M, ω) sympl

g, J (al. cx. str.) ω-compatible so $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$

$u : \mathbb{R} \times S^1 \to M$ smooth

energy $$E(u) = \int_{\mathbb{R}\times S^1} \underbrace{|\partial_s u|^2}_{= g(\partial_s u, \partial_s u) = \omega(\partial_s u, J\partial_s u)} ds\, dt$$

if $-\nabla A_H$ trajectory, so $\partial_s u + J(\partial_t u - X_H) = 0$:

$$E(u) = \int_{\mathbb{R}\times S^1} \omega(\partial_s u, \partial_t u - X_H)\, ds\, dt$$

$J\partial_s u = J(-J(\partial_t u - X_H)) = \partial_t u - X_H$ $J^2 = -\mathrm{id}$

$\omega(\cdot, X_H) = dH$

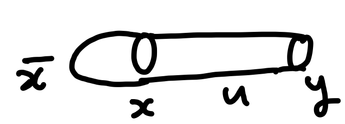
$$= \int_{\mathbb{R}\times S^1} \omega(\partial_s u, \partial_t u)\, ds \wedge dt - \int_{\mathbb{R}\times S^1} \underbrace{dH(\partial_s u)}_{= \partial_s(H \circ u)}\, ds\, dt$$

$$= \int_{\mathbb{R}\times S^1} u^*\omega - \int_{S^1} (H(y) - H(x))\, dt$$

$$= A_H(\bar{x}) - A_H(\bar{y})$$

$$A_H(\bar{x}) = -\int_{\bar{x}} u^*\omega + \int_{S^1} H(x)\, dt$$

capping $\bar{y}$ is

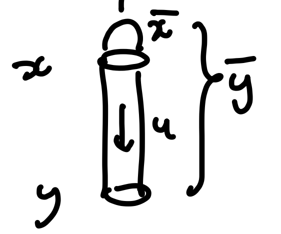


so difference in $u^*\omega$ integrals of A_H's is precisely $\int_u u^*\omega$

u is s-indep 1-orbit of X_H

$\Updownarrow$

Upshot: $E(u) \geq 0$ and $= 0 \iff \partial_s u \equiv 0 \iff \partial_t u - JX_H \equiv 0$



$E(u) = A_H(\bar{x}) - A_H(\bar{y})$ "a priori energy estimate"

not entirely a priori because it depends on filling $\bar{y}$ which depends on u !

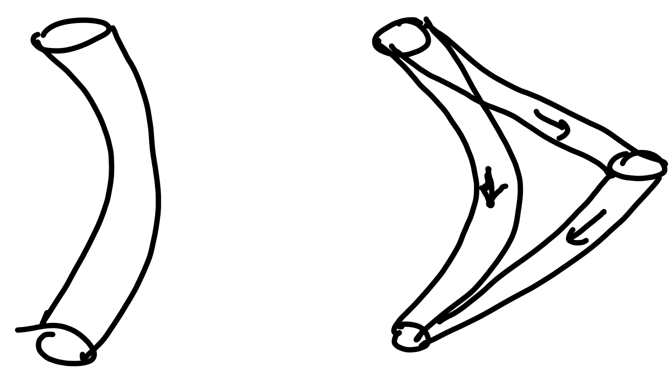
exercise show that $E(u)$ is a homotopy invariant of u relative to the ends x, y

Hint consider a 1-family $(u_\lambda)_{0 \leq \lambda \leq 1}$, use Stokes's theorem and fact $d\omega = 0$.

breaking picture:

e.g. proof of $\partial^2 = 0$

Will come back to next time (bubbling issue)



$\leftarrow$ energy is additive under breaking i.e. total energy is preserved like in Morse case

Next time Energy is also "quantised"

$\Rightarrow$ finite # possible breakings in given family (or in a sequence $u_n \to$ broken where assume a bound $E(u_n) \leq$ constant)