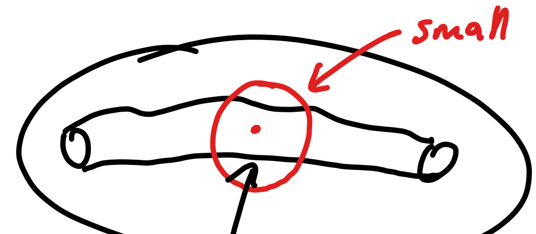
Morse 5

<u>Last time</u>: is energy "quantised" in Floer theory?

<u>Key idea</u> Monotonicity Lemma
(or isoperimetric inequality)

idea: $\Sigma \xrightarrow{\text{hol}}$  M

small ball $B(x,r)$

how much energy is consumed here? $E(u) = \int u^*\omega$?

<u>claim</u> $E \geq \text{constant} \cdot r^2$

<u>Rmk</u> $u : \Sigma \to M$ J-holo then (Riem surf.)

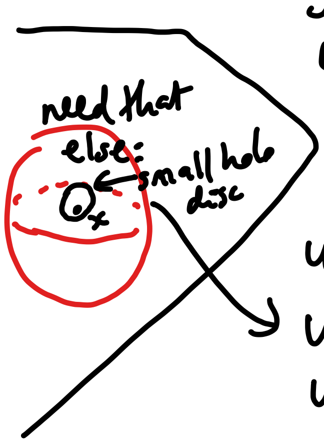
$$\int u^*\omega = \int \omega(\partial_s u, \partial_t u)\, ds\, dt = \int \omega(\partial_s u, J\partial_s u)\, ds\, dt = \int |\partial_s u|^2 ds\, dt = E(u)$$

$z = s + it$ local cx coord

J-holo : $du \circ j = J \circ du \iff \partial_s u + J \partial_t u = 0$

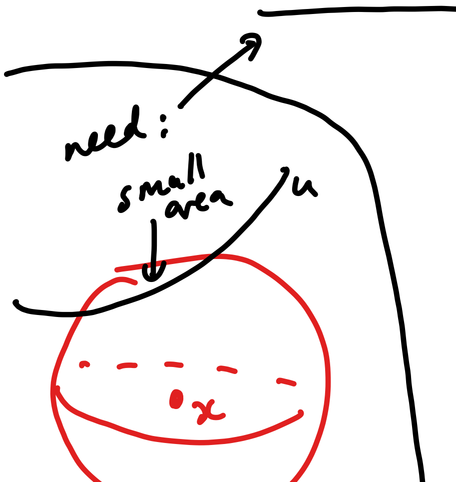
j cx str on Σ

<u>precise claim</u>



(M,ω) closed sympl mfd
J ω-compatible almost cx structure (so Riem. metric $g = \omega(\cdot, J\cdot)$)
$u : S \to M$ J-holo map (so $du \circ j = J \circ du$)
S Riemann surface with boundary ∂S ← in above picture a part of domain Σ that → into Ball

$u(S) \subseteq B := B(x,r)$ some $r \leq r_0$
$u(\partial S) \subseteq \partial B$
$u(S) \ni x$



$\exists\ r_0 > 0$, constant $c > 0$ such that

$$\boxed{E(u) \geq c \cdot r^2}$$

depend on an upper bound on sectional curvature, and lower bound on injectivity radius.

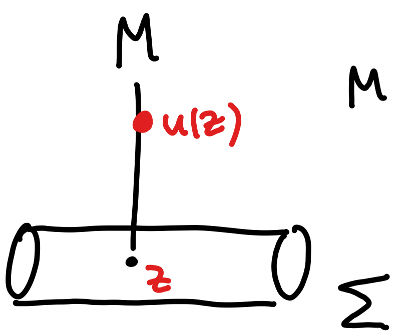
claim holds for noncompact M if "geometrically bounded" reference: Audin, Lafontaine "Holomorphic curves in symplectic geometry", Progr. Math. 117, Birkhäuser 1994

Sec. V. 4 (Sikorav)

because al. cx. str. rather than cx str.

Gromov (original paper on pseudoholo curves)

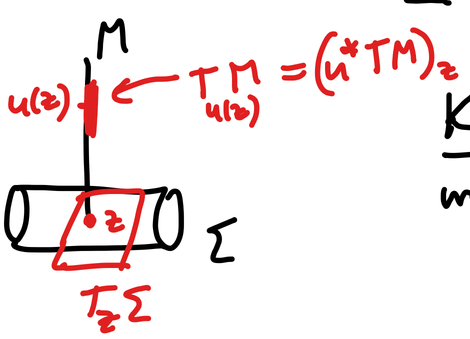
idea: turn the Floer eqn in M into a pseudo-holo eqn in bundle over Σ with fiber M



$M \times \Sigma$ in our case

$$M \times \Sigma$$
$$\downarrow$$
$$\Sigma$$

section $z \longmapsto \tilde{u}(z) = (z, u(z))$



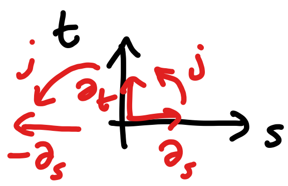
Key: more choice to build al. cx. str. $\tilde{J}$ on $M \times \Sigma$

we will push the Ham. part of Floer eqn into construction of $\tilde{J}$

section pseudoholo means $\quad d\tilde{u} \circ j = \tilde{J} \circ d\tilde{u}$

cx str on Σ

proof local cx coord $z = s + it$



Floer eqn: $\underbrace{\partial_s u}_{= du(\partial_s)} + J(\underbrace{\partial_t u}_{= du(\partial_t) = du \circ j(\partial_s)} - X_H) = 0$

can allow H, J to depend on z i.e. on s,t ⊛

rewrite: $\underbrace{(du + J \circ du \circ j)}_{\text{obstruction to being } J\text{-holo for } u}(\partial_s) = J X_H = (ds \otimes J X_H)(\partial_s)$

section of $T^*\Sigma$ $\downarrow$ Σ

$\Gamma(T^*\Sigma \otimes u^*TM)$

section of u^*TM $\downarrow$ $\Sigma \ni z$ $\quad T_{u(z)}M \downarrow z$

⊛ recall: need allow H_t time-dep. so that 1-periodic orbits are nondegenerate
also: when prove invariance of Floer cohomology under homotopies of H, J then allow s-dependence as well.

allow J_t time dep. to get transversality

$j(\partial_t) = -\partial_s$ since $j^2 = -id$

$$(du + J \circ du \circ j)(\partial_t) = \partial_t u - J \partial_s u = -J(J\partial_t u + \partial_s u) = -J^2 X_H$$
$$= X_H = (dt \otimes X_H)(\partial_t)$$

upshot: $\quad du + J \circ du \circ j = \nu := ds \otimes \underbrace{J X_H}_{\nu(\partial_s)} + dt \otimes \underbrace{X_H}_{\nu(\partial_t)}$

$$du + J\circ du \circ j = \nu := ds \otimes JX_H + dt \otimes X_H$$

<u>recall</u> section

$$\begin{tikzcd} \Sigma \times M \arrow[d] \\ \Sigma \arrow[u, bend right, "\tilde{u}"'] \end{tikzcd} \qquad \tilde{u}(z) = (z, u(z))$$

seek $\tilde{J}$ al. cx. str. on $\Sigma \times M$

$\tilde{u}$ $\tilde{J}$-holo $\Leftrightarrow \tilde{J}\circ d\tilde{u} = d\tilde{u}\circ j$

$$(\tilde{J}\circ d\tilde{u})(\partial_s) = \tilde{J}(\underline{\underline{\partial_s}}, \partial_s u)$$

$$\parallel$$

$$(d\tilde{u}\circ j)(\partial_s) = d\tilde{u}(\partial_t) = (\partial_t, \partial_t u)$$

$$= (\underline{\underline{j\partial_s}}, J\partial_s u + \underbrace{\nu(\partial_t)}_{= \nu\circ j(\partial_s)})$$

$\Rightarrow$ want $\tilde{J} := \begin{pmatrix} j & 0 \\ \nu\circ j & J \end{pmatrix}$ $\left[\cdot \begin{pmatrix} \partial_s \\ \partial_s u \end{pmatrix} = \begin{pmatrix} j\partial_s \\ \nu\circ j\partial_s + J\partial_s u \end{pmatrix} \right]$

<u>exercise</u> check $(\tilde{J}\circ d\tilde{u})(\partial_t) = (d\tilde{u}\circ j)(\partial_t)$ as well.

<u>Note</u>: $\tilde{J}^2 = \begin{pmatrix} j^2 & 0 \\ \underbrace{\nu\circ j^2 + J\nu j}_{= -\nu + J\nu j} & J^2 \end{pmatrix} = \begin{pmatrix} -id & 0 \\ 0 & -id \end{pmatrix}$ ✓

$$(-\nu + J\nu j)(\partial_s)$$
$$= -\nu(\partial_s) + J\nu(\partial_t)$$
$$= -JX_H + J\cdot X_H$$
$$= 0 \quad \text{similarly for } \partial_t$$

$$\tilde{J} : \underset{\substack{\parallel \\ T\Sigma \times TM}}{T(\Sigma \times M)} \to T(\Sigma \times M)$$

yes: $j : T\Sigma \to T\Sigma$ $\qquad$ $\begin{tikzcd} u^*TM \arrow[rr, "J"] \arrow[dr] & & u^*TM \arrow[dl] \\ & \Sigma & \end{tikzcd}$ $\quad J = J_z,\ z\in\Sigma$

ν section of $T^*\Sigma \otimes u^*TM$
so can view this as a
hom $T_z\Sigma \to T_{u(z)}M$
(the $T^*\Sigma$ eats the $T\Sigma$ input)

$\Rightarrow$ $\tilde{J}$ al. cx. str. on $\Sigma \times M$

<u>upshot</u> $u : \Sigma \to M$ Floer soln for H $\Longleftrightarrow$ $\tilde u : \Sigma \to \Sigma \times M$, $z \to (z, u(z))$

$\tilde J$- holo section

$$\tilde J = \begin{pmatrix} j & 0 \\ \nu \circ j & J \end{pmatrix}$$

$$\nu = ds \otimes J X_H + dt \otimes X_H$$

<u>sympl. form for $\Sigma \times M$ compatible with $\tilde J$?</u>

$\pi : \Sigma \times M \to \Sigma$

<u>claim</u> $$\tilde\omega := \overbrace{\omega - \underbrace{d(H\,dt)}}^{\tilde\Omega} + \underset{\text{some } c \gg 0}{c} \cdot \pi^* \underset{\text{area form for Riem surface }\Sigma\text{ (is sympl)}}{vol_\Sigma} \quad \text{works}$$

if H is s,t dependent: $\underbrace{dH}_{\text{differential form on } M} \wedge dt + \partial_s H \, ds \wedge dt$

<u>pf</u> $$\tilde\Omega(\underbrace{a\partial_s + b\partial_t}_{T\Sigma \text{ part}} + \underset{TM\text{ part}}{\vec m}, \; a'\partial_s + b'\partial_t + \vec m')$$

$$= \omega(\vec m, \vec m') - dH(\vec m)\, b' + dH(\vec m')\, b - \partial_s H \cdot (ab' - a'b)$$

$$= \omega(\vec m - b X_H, \; \vec m' - b' X_H) - (ab' - a'b)\partial_s H$$

$\omega(\cdot, X_H) = dH$
$\omega(X_H, \cdot) = -dH$

hence: $$\tilde\Omega(\underbrace{a\partial_s + b\partial_t + \vec m}_{v}, \; \underbrace{\tilde J(a\partial_s + b\partial_t + \vec m)}_{a\partial_t - b\partial_s + J\vec m + a\underbrace{\nu(\partial_t)}_{= X_H} - b\underbrace{\nu(\partial_s)}_{= J X_H}})$$ $\quad \tilde J = \begin{pmatrix} j & 0 \\ \nu \circ j & J \end{pmatrix}$

$/\!/$ $\tilde\Omega(v, \tilde J v)$ for general v

$$= \omega(\vec m - b X_H, \; \underbrace{J(\vec m - b X_H) + \cancel{a X_H}}_{\vec m'} - \underset{b' = a}{\cancel{a X_H}}) - (a^2 + b^2)\partial_s H$$

general formula above

don't know sign!

$$= \underbrace{\omega(\vec m - b X_H, \; J(\vec m - b X_H))}_{|\vec m + b X_H|_g^2 \geq 0} - \underbrace{(a^2 + b^2)\partial_s H}_{\text{sign looks dangerous}}$$

$$\tilde\omega(v,\tilde J v) = |\dots|_g^2 - \underbrace{(a^2+b^2)}_{|w|^2}\underbrace{\partial_s H}_{\substack{\text{bounded by}\\ \max_M |\partial_s H|}} + c\cdot \pi^*\mathrm{vol}(w, jw)$$

$w = a\partial_s + b\partial_t$

$\Rightarrow$ $c \gg 0$ then last term beats
($\uparrow$ large)

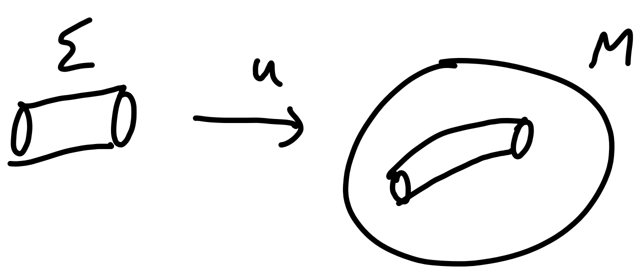
$\Rightarrow$ $\tilde\omega(v,\tilde J v) \geq 0$
$= 0$ iff $v = 0$ $\boxed{M}$

always: COMPACT SUPPORT for z-dependence in H_z, J_z

crucially: if Σ noncompact like $\mathbb{R}\times S^1$ then only allow s-dependence on a compact subset

(OK e.g. if do hty $H_0 \rightsquigarrow H_1$
$(H_\lambda)_{\lambda\in[0,1]}$
then extend
$$H_s = \begin{cases} H_0 & s\leq 0 \\ H_1 & s\geq 1 \\ H_\lambda & 0\leq\lambda\leq 1\end{cases}$$
$\mathbb{R}\times S^1$ coord $s\in\mathbb{R}$

Gromov:



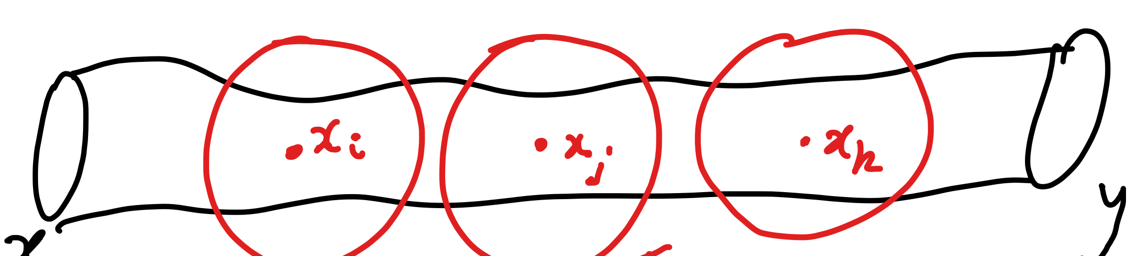
$\rightsquigarrow$ $\tilde J$-hol section

$$\begin{tikzcd} M\times\Sigma \arrow[d] \\ \Sigma \arrow[u, bend left, "\tilde u"] \end{tikzcd}$$

$\rightsquigarrow$ monotonicity lemma for $\tilde J$-hol map
$\tilde u : \Sigma \to M\times\Sigma$
applies!

$\Rightarrow$ suppose distinct 1-orbits x, y for X_H

assume 1-orbits of X_H are isolated (perturb H if necessary)



$B(x_i; r)$ $\leftarrow r < r_0$

on each ball $u \cap$ ball consumes energy $\geq$ const $\cdot\, r^2$

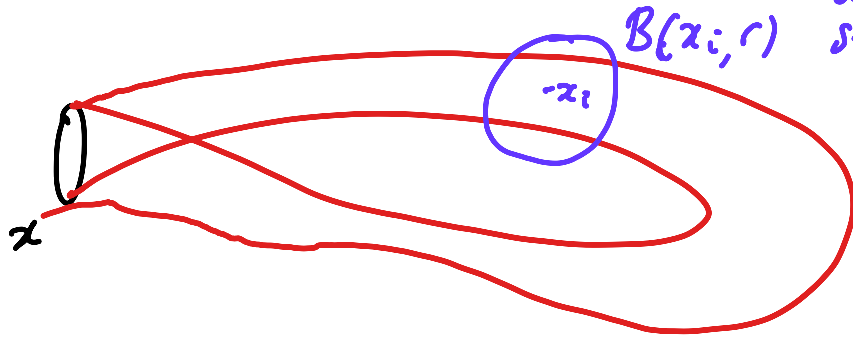
depends on M not u

$$\Rightarrow \boxed{E(\text{Floer solution}) \geq \hbar}$$

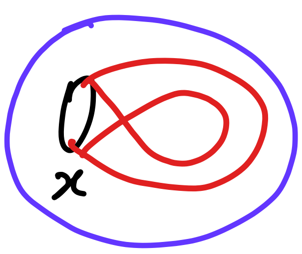
e.g. const. r^2
suff small r

e.g. $r < \frac{\min}{100}\left(r_0,\ \text{distance between distinct 1-orbits}\right)$

What if some 1-orbit?



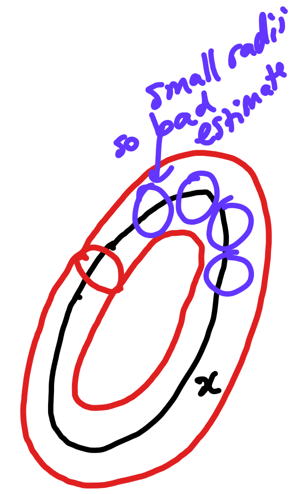
remaining problem:



← happening in arbitrarily small ball or small nbhd of 1 orbit

issue: constant $\cdot\, r^2$ ← arb. small

So don't get $\hbar$



Last Trick: pick small tubular neighbourhood of 1-orbit

$\Rightarrow$ homology of solid torus

$\Rightarrow H^2(\text{solid torus}) \cong H^2(\text{circle}) = 0$

$\Rightarrow [\omega]\big|_{\text{solid torus}}$ is exact!

$\Rightarrow A_H$ is well-defined without using "caps":

$$A_H(x) = -\int_{S^1} x^*\theta + \int_{S^1} H(x)\, dt$$

$\Rightarrow E(u) = A_H(x) - A_H(x) = 0$ ← since u has equal asymptotics $\Longrightarrow \partial_s u = 0 \Rightarrow \partial_t u = X_H \Rightarrow$ u s-indep. 1-orbit.

Upshot: $E\left(\begin{array}{l}\text{Floer solution that is} \\ \underline{\text{not}}\ s\text{-independent 1-orbit}\end{array}\right) \geq \text{some } \hbar > 0$

"energy quantisation" ✓

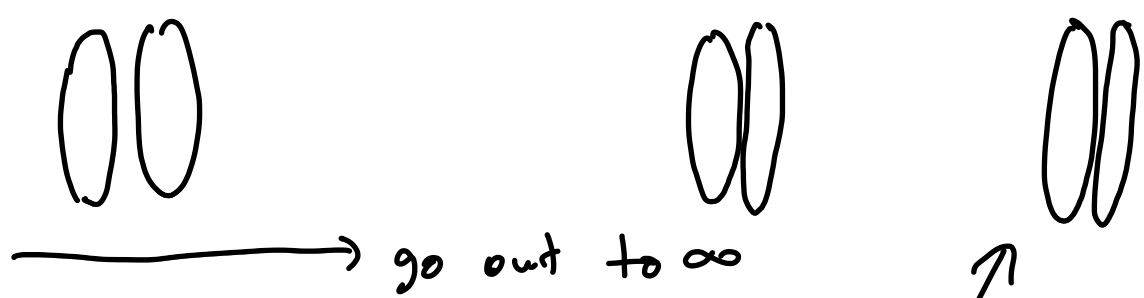
Rmk The argument assumes we pertubed H so that 1-orbits of X_H are "nondegenerate"

consequence of this: 1-orbits are isolated

since assume M closed (so compact): finitely many 1-orbits

distinct 1-orbits are a strictly positive distance apart

Non compact case can be dangerous:



still nondegenerate
but orbits may get closer & closer

so a priori could have very low energy Floer solns between them.

Exact $\omega \Rightarrow$ closed holo curves are constant

<u>Claim</u> $u: S \to M$ J-holo, and lands in a region where ω is exact, then u is constant.

S <u>closed</u> Riem surface

$\partial S = \emptyset$

<u>Pf</u> $\omega = d\theta$ on image of u

Stokes's Thm

$$E(u) = \int_S u^*\omega = \int_S \underbrace{u^* d\theta}_{= d(u^*\theta)} = \int_{\partial S = \emptyset} \theta = 0$$

$$E(u) = \int |\partial_s u|^2 \, ds\, dt \quad \text{in local coords}$$

$\Rightarrow$ $\partial_s u \equiv 0$ so $\partial_t u = J\partial_s u = 0$ so u constant ∎

<u>Next time</u>:
- bubbling
- invariance proof in Morse theory &
- $QH^*(M) \cong HF^*(M, H)$