Compactness of Moduli spaces in Morse theory & Floer theory

ODE theory M closed mfd
V smooth vector field

$\Rightarrow$ flowline u (so solution of $u'(s) = V|_{u(s)}$)
- exists for all time $s \in \mathbb{R}$
- uniquely determined by initial condition $p = u(0) \in M$
- depends smoothly on " "

can fail if M noncompact

Rmk uniqueness $\Rightarrow$ if flowlines intersect then they equal after $\mathbb{R}$-reparametrisation $\leftarrow u(\cdot + r)$ $\leftarrow$ constant

Rmk $V := -\nabla f$ $\quad f : M \to \mathbb{R}$ Morse

uniqueness $\Rightarrow$ takes ∞ time to reach a critical pt

$\longrightarrow \bullet$ $p \in \mathrm{Crit}\, f$ $\leftarrow$ unique flowline = constant

$\uparrow$ f decreases along flowlines:

$$\begin{aligned} \partial_s (f \circ u) &= df \cdot \partial_s u \\ &= g(\nabla f, \partial_s u) \\ &= g(\nabla f, -\nabla f) \\ &= -|\nabla f|^2 \\ &\leq 0 \quad \text{and} < 0 \text{ unless at critical pt} \end{aligned}$$

$p \in \mathrm{Crit}\, f \Leftrightarrow d_p f = 0 \Leftrightarrow \nabla f|_p = 0$

Compactness theorem

$$\mathcal{M}(p,q) = W(p,q) / \mathbb{R}$$

$\uparrow$ moduli space of "unparametrised" flowlines or "trajectories"

$\uparrow$ $u(\cdot + r)$

$W(p,q) = \mathcal{U}(p) \cap \mathcal{D}(q)$

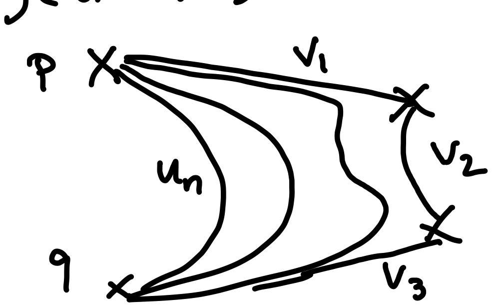
$\uparrow$ u with $u \to p$ as $s \to -\infty$

$\uparrow$ u with $u \to q$ as $s \to +\infty$

$= \{ u : \mathbb{R} \to M : u' = -\nabla f, \ u \to p, q \text{ at } -\infty, +\infty \}$

"parametrised flowlines"

is compact up to "broken trajectories"



$\leftarrow$ finitely many flowlines v_i

Preliminary:

Arzelà–Ascoli Theorem

K compact metric space

$F \subseteq C(K) = \{\text{continuous } f: K \to \mathbb{R}\}$

family is equibounded ← $\sup_{x\in K} |f(x)| < \infty$ for each $f \in F$

and equicontinuous ← $\forall \varepsilon > 0,\ x \in K$ $\exists$ nbhd $U \ni x$ with $|f(y) - f(x)| < \varepsilon$ $\forall y \in U$ $\forall f \in F$

$\Rightarrow \overline{F} \subseteq C(K)$ is compact.

closure

Explicitly: $\forall f_n \in F$ $\Rightarrow \exists$ unifly cgt subsequence ← limit need not be in F.

How used in case of manifolds M?

fact $M \xhookrightarrow{} \mathbb{R}^N$ ← large

$\exists$ embedding

← Whitney embedding thm

If choose Riemannian metric g on M, then can make the embedding isometric (Nash embedding thm)

If M closed mfd, $\Sigma \xrightarrow{u} M \xhookrightarrow{} \mathbb{R}^N$

any mfd, any cts map, compact ⟶ compact so bounded

$\Rightarrow$ automatically equibounded since

if $\Sigma \xrightarrow{u} M$ is smooth (or C^1) and $\boxed{du \text{ is bdd}}$ then the Mean Value Theorem gives equicontinuity for free

1-dim case: $|f(y) - f(x)| \leq \underbrace{|f'(c)|}_{} \cdot |y-x|$

some $c \in (x,y)$ if bdd then get equicty

$\exists$ similar MVT in $\mathbb{R}^n$

Upshot du_n bdd $\Rightarrow$ Arzela–Ascoli applies: $\exists$ subsequence of u_n which cge uniformly on compact sets

($u_n \in$ some family F of such $u: \Sigma \to M$)

use $K \subseteq \Sigma$ compact above

called C^0_{loc}-convergence

WORK WITH VECTOR FIELD $V = -\nabla f$, $f: M \to \mathbb{R}$ ← closed mfd

(We chose a Riemannian metric g on M and ∇f defined by $g(\nabla f, \cdot) = df$
& define $|w| = \sqrt{g(w,w)}$ for tangent vectors w in M.)

<u>Claim</u> u_n flowlines $\Rightarrow$ $\exists$ subseq that cges in C^0_{loc}

<u>pf</u> $u_n' = -\nabla f \Rightarrow |u_n'| \leq \max_M |\nabla f| \Longrightarrow$ Arzelà–Ascoli ∎

<u>Cor</u> $W := \bigcup_{p,q \in \mathrm{Crit} f} W(p,q)$ is C^0_{loc}-compact

← need union since have not proved (indeed false) that limit has same crit endpoints.

<u>pf</u> Need explain why limit u of subsequence u_n is a flowline:

all we know: $u: \mathbb{R} \to M$ is continuous

<u>"ELLIPTIC BOOTSTRAPPING"</u>

f smooth

$u_n \to u$ in $C^0_{loc} \Longrightarrow -\nabla f|_{u_n} \longrightarrow -\nabla f|_u$ in C^0_{loc}

$\Longrightarrow u_n' \longrightarrow u'$ in C^0_{loc}

$\exists u'$: in chart

$$u_n(x+h) = u_n(x) - \nabla f|_{u_n} \cdot h + o(h)$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$u(x+h) = u(x) - \nabla f|_u \cdot h + o(h)$$

$\Rightarrow u_n \to u$ in C^1_{loc}

Now repeat argument to get $C^2_{loc}, C^3_{loc}, \ldots$

This is called $\boxed{C^\infty_{loc}}$ cgce.

($\forall$ compact, $\forall k \in \mathbb{N}$ the kth deriv cges)

Upshot u is smooth (smoothness is local condition) and $u' = -\nabla f$. ∎

corollary:

$\boxed{\text{Prop 1}}$ W is C^∞_{loc}-compact.

Why C^∞_{loc} is not good enough as a topology on W

- C^∞ / C^∞_{loc} is not a Banach manifold topology

  like mfd, but charts are modelled on Banach spaces, rather than finite dim vector space ($\mathbb{R}^n$)
  So allow ∞-dim vectorspace.

- C^∞_{loc} does <u>not</u> allow us to define energy in general

  $$E(u) = \int_{\mathbb{R}} |\partial_s u|^2 \, ds \quad \text{need not be finite}$$

  $u : \mathbb{R} \to M$ smooth

  would need to impose L^2

- Why want Banach mfd topology? Transversality Theorem:

$$\begin{tikzcd}
\{\text{vector fields along those "paths"}\} \arrow[d, "\text{infinite dim vector bdle}"'] \\
\{\text{"paths" } \mathbb{R} \to M \text{ from } p \text{ to } q\} \arrow[u, bend right, "F"']
\end{tikzcd} \qquad F(u) = \partial_s u - \nabla f|_u$$

$$W(p,q) = (F\text{section}) \cap (0\text{ section})$$
$$= F^{-1}(0)$$

by picking metric g generically we hope this is a TRANSVERSE INTERSECTION so hope it is a smooth mfd

to prove this need <u>implicit function theorem</u> applied to a <u>Fredholm section</u> of a <u>Banach vector bundle</u>

at its heart it's Banach fixed pt thm, which needs completeness

Fredholm map means differential is <u>Fredholm operator</u>

Means linear operator of Banach spaces with Ker & Coker both finite dimensional.

Intuition: up to finite dimensional errors these are isomorphisms.

Good choice of topology:

$W^{1,2}_{loc}(\mathbb{R}, M)$

so $W^{1,2}$ on compact sets

completion of smooth maps w.r.t. inner product

$\langle u,v\rangle_{L^2} + \langle u', v'\rangle_{L^2}$

$u,v : \mathbb{R} \to \mathbb{R}^N$

$\langle u,v\rangle_{L^2} = \int_{\mathbb{R}} u \cdot v$ dot prod. in $\mathbb{R}^N$

Need "loc"

$W^{1,2}(\mathbb{R}, \mathbb{R}^N)$ is terrible:

$f: \mathbb{R} \to M \to \mathbb{R}^N$

if M lands in $\mathbb{R}^N \setminus \{0\}$ then all such f have $\int |f| = \infty$!

Topology on W

- $u: \mathbb{R} \to M$ is $W^{1,2}_{loc}(\mathbb{R}, M)$
- some large $r \in \mathbb{R}$

  $u|_{(-\infty,-r)}$ is $W^{1,2}$ in a chart around $p = u(-\infty)$ via exponential map

  $T_pM \ni v$ geodesic for time 1

  p $\exp_p(v)$

  $\exp_p : (\text{small nbhd } 0 \in T_pM) \xrightarrow{\cong} (\text{nbhd of } p \in M)$

  $u(s) = \exp_p(\xi(s))$

  $\xi : (-\infty, -r) \to T_pM \cong \mathbb{R}^n$ is $W^{1,2}$

  so $\xi \in L^2$, $\xi' \in L^2$

- $u|_{(r,\infty)}$ is $W^{1,2}$ in exp-chart around $q = u(+\infty)$.

In transversality thm we work with

$\{L^2\text{-vector fields along such } u\}$

$\downarrow$

$\{u : \mathbb{R} \to M \text{ with above topology from } p \text{ to } q\}$

$F(u) = \partial_s u - \nabla f|_u$

we lost one derivative

the "$W^{1,2}$" topology

Sobolev embedding thm

recall: can change L^p functions on set of measure 0, for $W^{k,p}$ spaces $\exists$ the danger that $\nexists$ cts representative

$$W^{1,2}(\mathbb{R},\mathbb{R}^N) \underset{\substack{\text{embedding} \\ \text{(and bdd linear map)}}}{\hookrightarrow} C^0_{bdd}(\mathbb{R},\mathbb{R}^N)$$

← cts bdd maps

can replace $\mathbb{R}$ by any open set $S \subseteq \mathbb{R}$ with smooth ∂S.

(For $W^{k,p}$ spaces, need $k \cdot p >$ dim domain = dim $\mathbb{R} = 1$)

<u>Rmk</u> headache in Floer theory : domain $\mathbb{R}\times S^1$ is dim=2

so need $kp > 2$

so cannot use $W^{1,2}$, need use $W^{1,p}$ $p>2$

↑ very nice: Hilbert space (have inner product)

Rellich Theorem

$$W^{1,2}(\mathbb{R},\mathbb{R}^N) \xrightarrow{\text{restrict}} W^{1,2}(\underbrace{(-r,r)}_{\text{now bounded}},\mathbb{R}^N) \xrightarrow[\text{EMBEDDING}]{\text{COMPACT}} C^0([-r,r],\mathbb{R}^N)$$

image of bounded sets have compact closure

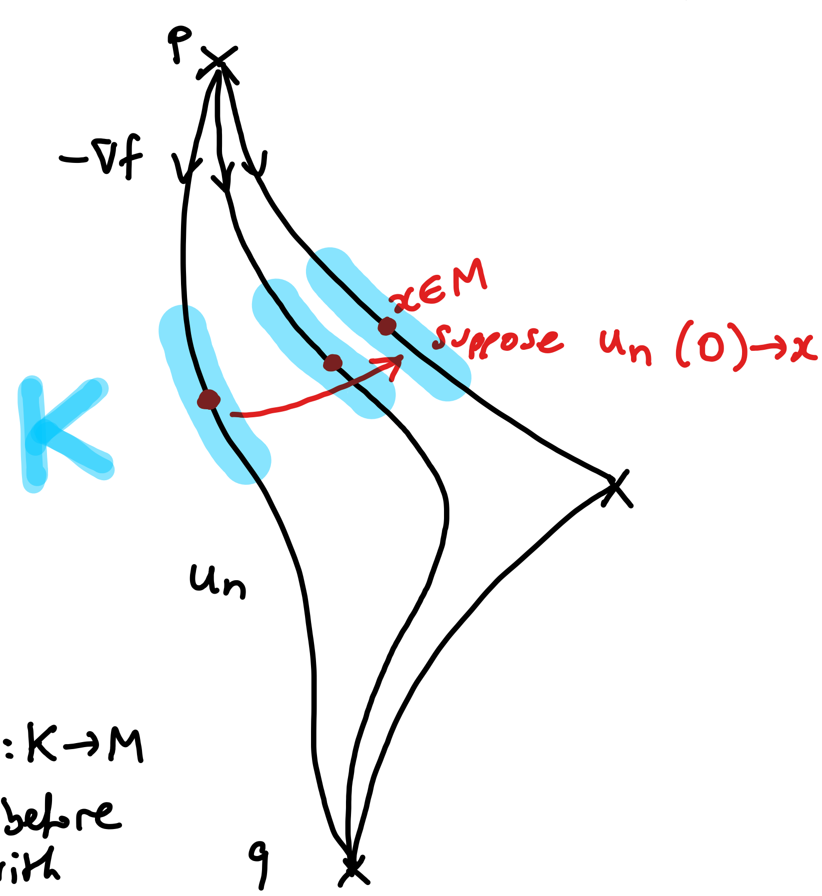
Consequence: $W^{1,2}_{loc}$ cgce $\Rightarrow$ C^0_{loc} cgce.

<u>Def</u> Topology on $\mathcal{M}(p,q) = W(p,q)/\mathbb{R}$ is quotient topology of that "$W^{1,2}$" top.

so : $[u_n] \longrightarrow [u]$ means $\exists\, r_n \in \mathbb{R}$

$u_n(\cdot + r_n) \longrightarrow u(\cdot)$ in "$W^{1,2}$" top on $W(p,q)$

<u>Intuition</u>:



on $K = [-r,r]$ compact

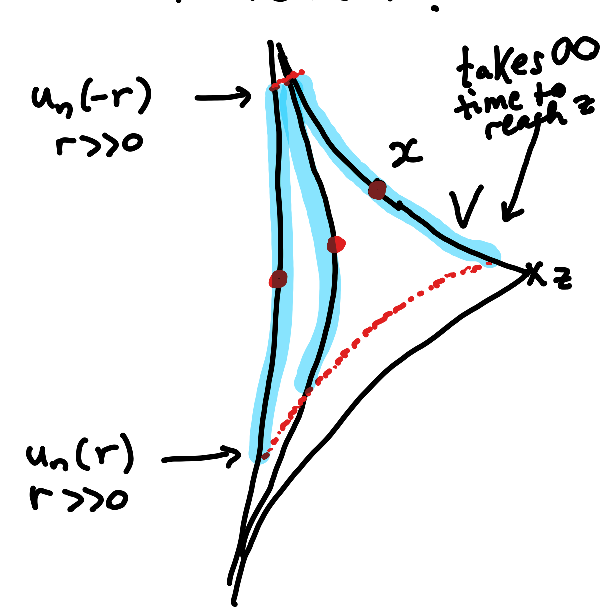
$u_n|_K \to$ some $v : K \to M$

by ODE result from before

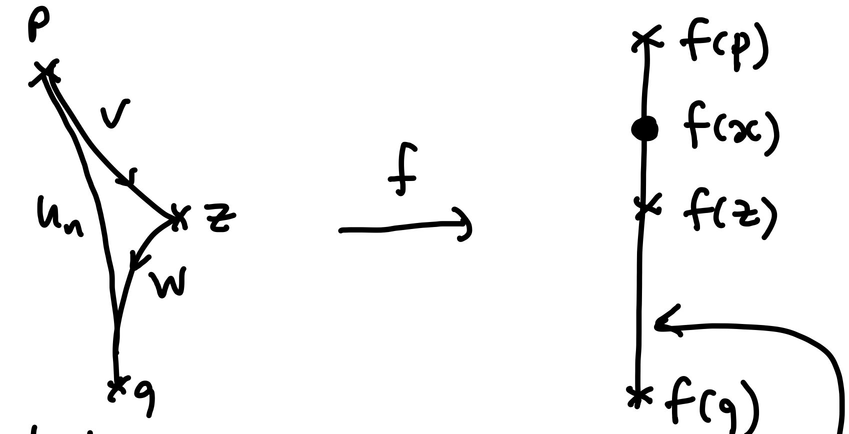
v is unique flowline with

$v(0) = x$

What happens if increase r?



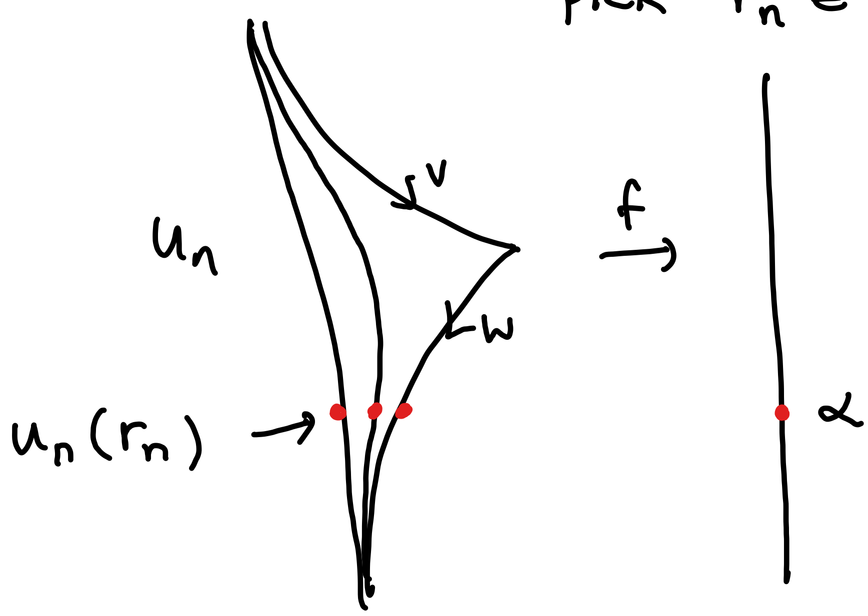
So the $u_n(\cdot) \to v(\cdot)$ C^∞_{loc} and do not seem to detect w



Only when you reparametrise, you detect w:

<u>REPARAMETRISATION TRICK</u> pick value $\alpha \in (f(q), f(z))$

pick $r_n \in \mathbb{R}$: $f(u_n(r_n)) \to \alpha$



note $f(u_n(\mathbb{R})) \overset{\text{cty}}{=} (f(q), f(p)) \subseteq \mathbb{R}$ using that f decreases along flowlines

Again by ODE or Arzelà-Ascoli : $u_n(\cdot + r_n) \xrightarrow{C^\infty_{loc}} w(\cdot)$

so detect broken pieces by reparametrisation

<u>Exercise</u> Use this trick (uniqueness & smooth dependence on initial conditions of ODE solns) to prove Compactness theorem, using less machinery

<u>Rmk</u> Reason for showing more analytical proof : in Floer theory case the flow of $-A_H$ on $\mathcal{L}M$ is ill-defined : cannot use ODE tricks.

eg. flow is not even defined for short time!

Warnings

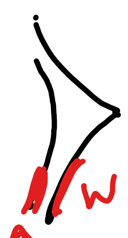
- $u_n \in W = \bigcup_{p,q} W(p,q)$, $u_n \in W(p,q)$ $\Longrightarrow$ subseq. $u_n \to u$ C^0_{loc} cging in W
  (u: need not be in $W(p,q)$)

  if $u \notin W(p,q)$ then
  $[u_n] \not\to [u]$ in $\mathcal{M}(p,q)$

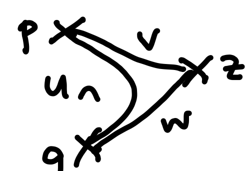
  indeed expect $[u_n] \longrightarrow$ "broken trajectory" $\in$ "$\partial\mathcal{M}(p,q)$"
  in $M(p,q)$ $\longrightarrow$ $\;= [u_n(\cdot + r_n)]$

  new points you add to compactify $\mathcal{M}(p,q)$

- Energy "escapes to ∞" in this case



e.g.



$u_n \to v$
$u_n(\cdot + r_n) \to w$

then $r_n \to \infty$ (since v needs ∞ time to reach crit. pt. z)

idea: $E(u_n|_{s \gg 0}) \approx E(w)$

careful: $E(u_n(\cdot)) \not\to E(v)$
$\quad \| $
$E(u_n(\cdot + r_n)) \not\to E(w)$
$\quad \|$
$f(p) - f(q)$

$$
\begin{aligned}
E(v) + E(w) &= (f(p) - f(z)) + (f(z) - f(q)) \\
&= f(p) - f(q) \\
&= E(u_n)
\end{aligned}
$$

$\boxed{\text{Prop 2}}$ ← "no breaking case"

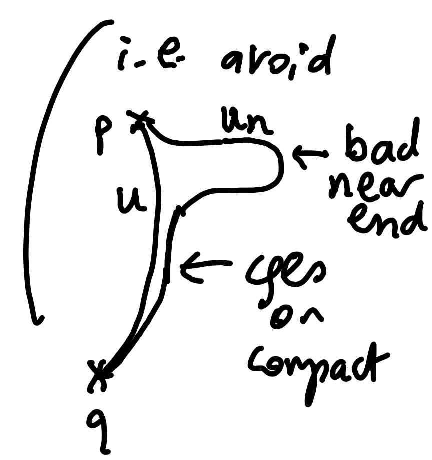
$u_n \to u$ C^0_{loc} in $\underline{\underline{W(p,q)}}$ $\Longrightarrow$ $u_n \to u$ in "$W^{1,2}$" top. of $W(p,q)$ $\Rightarrow$ $[u_n] \to [u]$ in $M(p,q)$

(so assume p,q are ends of u as well)

Pf $\quad u_n \to u$ in C^0 on $[-r, r]$ (assumption)

$\Longrightarrow$ (elliptic bootstrapping) cges C^∞ on $[-r, r]$ so $W^{1,2}_{loc}$ cgce ✓

Need prove $W^{1,2}$ cgce at ends in charts



__Trick__ Use energy!

$$E(u_n) = f(p) - f(q) = E(u)$$

$\Rightarrow E(u|_{(-\infty, -r]})$ is small for sufficiently large r.
and similarly for $u|_{[r, +\infty)}$.

$\leftarrow E(u|_{[-r,r]}) \approx E(u)$ for $r \gg 0$

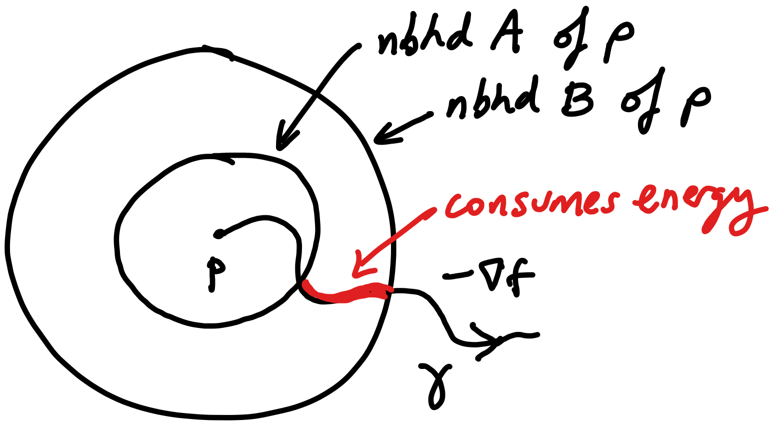
$\Rightarrow E(u_n|_{(-\infty, -r]})$ small as $n \to \infty$ $\quad$ so "no energy escapes to ∞"

because
- $u_n \to u$ in C^∞ on $[-r, r]$ so $E(u_n|_{[-r,r]}) \to E(u|_{[-r,r]})$
- $E(u_n) = E(u)$

<u>Upshot</u> $E(u_n|_{(-\infty,r]})$, $E(u|_{(-\infty,r]})$ are small

<u>Consequence</u> $u_n|_{(-\infty,r]}$, $u|_{[-\infty,r]}$ are trapped in an arbitrarily small nbhd of p for large r. $\left(p = u(-\infty) = u_n(-\infty)\right)$

<u>Pf</u> <u>"NO ESCAPE LEMMA"</u>



$\overline{A} \subseteq B$, $\quad \mathrm{Crit}\, f \cap B = \{p\}$

$\exists\, \delta > 0$ s.t. $\forall$ $-\nabla f$ flowline $\gamma : (a,b) \to M$ from ∂A to ∂B (or viceversa) has $\quad E(\gamma) \geq \delta \cdot \mathrm{dist}(\partial A, \partial B)$

<u>Pf</u> $|\nabla f| \neq 0$ on $B \setminus A^\circ$, $\quad \delta := \left(\min\limits_{B\setminus A^\circ} |\nabla f|\right)^{1/2}$

$$E(\gamma) = \int |\partial_s \gamma|^2 = \int \underbrace{|\partial_s\gamma|}_{= -\nabla f} \cdot |\partial_s \gamma| \geq \delta \int_a^b |\partial_s \gamma| = \delta \cdot \mathrm{length}(\gamma). \quad \blacksquare$$

<u>Final ingredient</u> about how gradient flows in $\mathbb{R}^n$ cge to crit. pt. (more generally: hyperbolic flows)

<u>Fact</u> : <u>"EXPONENTIAL CONVERGENCE AT ENDS"</u>

near p in a chart: $\quad u(s) = \exp_p(\xi(s))$

$$\partial_s u = -\nabla f \implies |\xi(s)| \leq c \cdot e^{\delta \cdot s} \quad \forall s < r \qquad (\star)$$

constants $c, \delta > 0$ independent of u

<u>Consequence</u>

$u_n \to p$, $u \to p$ exponentially fast as $s \to -\infty$

$\implies L^2$-cgce in chart near p : $\quad u_n(s) = \exp_p(\xi_n(s))$, $\ u(s) = \exp_p(\xi(s))$

① By $(\star)$: $\int_{s \leq -R} |\xi_n|^2$, $\int_{s\leq -R} |\xi|^2 < \varepsilon$ for $R \gg 0$. $\quad$ (given $\varepsilon > 0$ $\exists R \ldots$)

② Moreover $\int_{-R \leq s \leq -r} |\xi_n - \xi|^2$ as $n \to \infty$ by C^0_{loc} cgce

So: $\left(\int_{s \leq -r} |\xi_n - \xi|^2\right)^{\frac{1}{2}} \leq 2\varepsilon + \ldots \to 2\varepsilon$ as $n \to \infty$

by ① using triangle inequality for L^2-

As $\varepsilon > 0$ was arbitrary, $\|\xi_n - \xi\|_{L^2} \to 0$ ✓

<u>Exercise</u> Run a similar argument to show $\|\xi_n' - \xi'\|_{L^2} \to 0$, hence

$$\xi_n \to \xi \quad \text{in} \quad W^{1,2}$$

so

$W^{1,2}$ cgce $u_n \to u$ near p

Similarly " " " " q $\quad (s \to +\infty)$. ∎

Hint. $u_n' = \nabla f|_{u_n}$, $u_n = \nabla f|_u$, both $\to \nabla f|_q = 0$ at $-\infty$

Compactness Thm

$[u_n] \in M(p,q) \Rightarrow \exists$ subseq. u_n which:

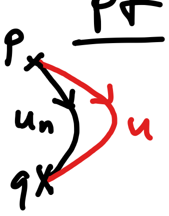
- either cges in $M(p,q)$
- or $\exists$ $-\nabla f$ trajectories $v_1, \dots, v_k$ ← some finite $k \geq 2$

  $\exists$ reparametrisation constants $r_{1,n}, \dots, r_{k,n}$

such that

$$u_n(\cdot + r_{j,n}) \to v_j \quad \text{as } n \to \infty$$

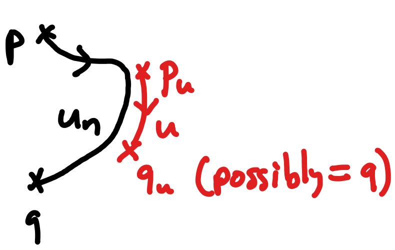
in C^∞_{loc}, and in "$W^{1,2}$" sense in W.



<u>pf</u> Prop 1 $\Rightarrow$ $u_n \to u$ in W in C^∞_{loc}

Case 1: $u \in W(p,q)$ $\xRightarrow{\text{Prop 2}}$ $u_n \to u$ in $M(p,q)$

Case 2: $u \notin W(p,q)$ $\Rightarrow$ use REPARAMETRISATION TRICK



So: $\quad u \in W(p_u, q_u) \qquad f(p) \geq f(p_u) \geq f(q_u) \geq f(q)$

say WLOG $\quad p_u \neq p$ so this strict ⟸ use f decreases strictly on nonconstant $-\nabla f$ flowlines

pick $\alpha \in (f(p), f(p_u))$

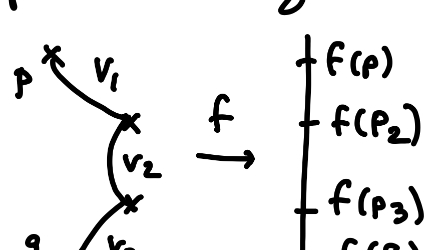
r_n s.t. $f(u_n(r_n)) = \alpha$

let $\tilde{u}_n := u_n(\cdot + r_n)$

and repeat the argument... ← note $f(\tilde{u}_n(0)) = \alpha$ so $\tilde{u}_n \not\to u$

$\Rightarrow u_n(\cdot + r_{j,n}) \to v_j$

By energy quantisation, only finite # of breakings occur.



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