Compactness theorems in Floer theory versus Morse case

very nice notes on Morse & Floer theory:
⤳ Salamon, Lectures on Floer homology
(on author's website
or "Park City" book)

Arzelá-Ascoli arguments are very similar:

$$u : \mathbb{R} \times S^1 \longrightarrow M \leftarrow (M,\omega) \text{ closed symplectic manifold}$$

or:
$u : \mathbb{R} \to \mathcal{L}M = C^\infty(S^1, M)$
$\partial_s u = -\nabla A_H$
"action functional"

Floer's equation

$$\partial_s u + J(\partial_t u - X_H) = 0$$

ω-compatible almost complex structure
(so $g := \omega(\cdot, J\cdot)$ Riem. metric)

$H : M \to \mathbb{R}$ "Hamiltonian"
($\omega(\cdot, X_H) = dH$
or: $X_H = J\nabla H$)

RmK
This is an elliptic PDE (very nice)

RmK 2
"$\partial_s u + J\partial_t u$" part is a Cauchy-Riemann operator (ie. J-holo curve equation)
And "JX_H" is 0^{th} order perturbation.

Energy: $E(u) = \int_{\mathbb{R}\times S^1} |\partial_s u|^2 \, ds\, dt$

Recall we talked about a priori energy estimates.

$$\boxed{\text{Fact: A solution } u \text{ as above cges to 1-orbits of } X_H \iff E(u) < \infty.}$$

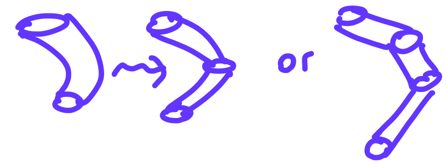
proof is Floer analogue of "exponential cgce at the ends"

exercise: in Morse case this is quite easy, show that flowlines → crit pts always.

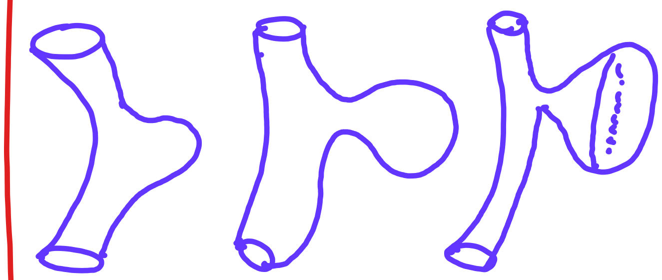
in Morse case, we got this automatically
$|du| = |\partial_s u| = |\nabla f| \leq \max_M |\nabla f|$

Key observation: if $|du|$ is bounded
then Arzelá-Ascoli applies (on compact subsets $K \subseteq \mathbb{R}\times S^1$)
so get C^0_{loc}-cgce
By elliptic-bootstrapping (since elliptic operator)
get C^∞_{loc}-cgce

Analogue of Morse breaking:



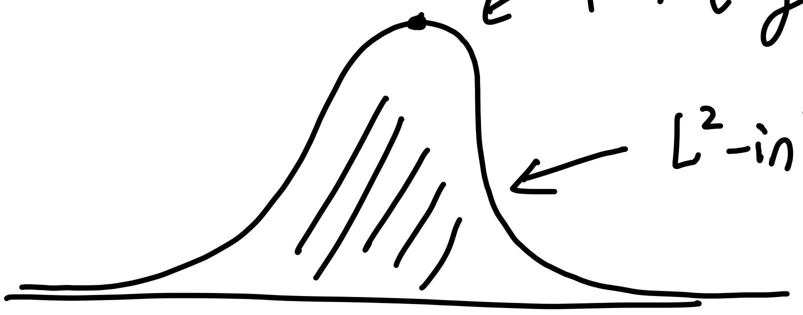
overview: showing a new "lack of cgce" issue different from breaking, called BUBBLING



"limit" in the compactified $\overline{\mathcal{M}(p,q)}$ is  holo sphere

$\Rightarrow$ New problem:

$|du|$ large



L^2-integral $|du|$ bdd

$\Rightarrow$ Arzelà-Ascoli trick can fail on a compact set if $\exists\, z_n \in \mathbb{R}\times S^1$ with

$$du_n\big|_{z_n} \longrightarrow \infty$$

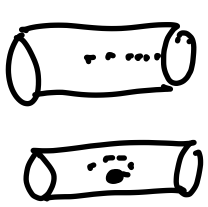
$\leftarrow$ SUPPOSE THIS HAPPENS (if not, get C^∞_{loc}-cgce like in Morse case)

sequence $u_n : \mathbb{R}\times S^1 \to M$ of Floer trajectories

<u>Rmk</u> z_n could diverge to $\pm\infty$ but can reparametrize $\tilde{u}_n(\cdot,\cdot) = u_n(\cdot - s_n, \cdot)$ so that (after taking subsequence)

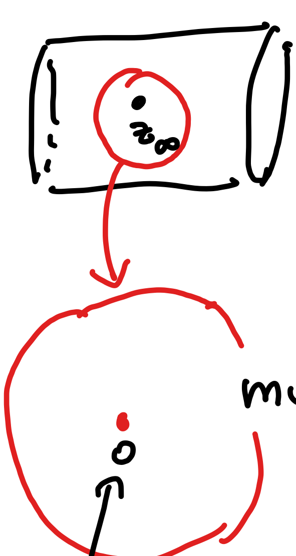
$$z_n \longrightarrow z_\infty \in \mathbb{R}\times S^1$$

$\mathbb{R}$-action on $\mathbb{R}$ coord of $\mathbb{R}\times S^1$ preserves Floer eqn



Key idea is a <u>rescaling trick</u>

rescale local complex coordinate z near $z_\infty \in \mathbb{R}\times S^1$



in local coords $z_\infty = 0$

$$v_n(z) := u_n(\varepsilon_n z)$$

rescaling constants $\varepsilon_n > 0$

notice:

$$\partial_s u_n + J(\partial_t u_n - X_H) = 0$$

multiply by ε_n

$$\underbrace{\varepsilon_n \partial_s u_n}_{=\partial_s v_n} + J(\underbrace{\varepsilon_n \partial_t u_n}_{=\partial_t v_n} - \underbrace{\varepsilon_n X_H}_{\to 0 \text{ if pick } \varepsilon_n \to 0}) = 0$$

<u>Upshot</u>: if v_n cges smoothly to v_∞ then $\partial_s v_\infty + J\partial_t v_\infty = 0$ is J-holo eqn!

How to force cgce of v_n?

need $|dv_n|$ bdd so that Arzelà–Ascoli applies

$$dv_n\Big|_z = \varepsilon_n \, du_n\Big|_{\varepsilon_n z}$$

so just pick for example

$$\varepsilon_n := \frac{1}{\left|du_n\big|_{z_\infty}\right|}$$

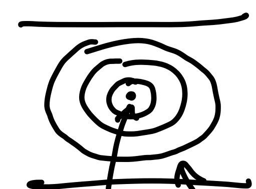
(omitting some details about precise size of nbhd of z_∞ : see Salamon's notes)

<u>Upshot</u>: $v_\infty : \mathbb{C} \longrightarrow M$

(fact that nbhd of z_∞ increased to $\mathbb{C}$ is part of details)

In Floer eqn, have $J\big|_z$ ← varying in $\mathbb{R}\times S^1$ because make J domain dependent to get transversality

but we are zooming into a nbhd of z_∞



z_∞ smaller nbhds as $n\to\infty$

In limit, it becomes

$$J\big|_{z_\infty}$$

independent of the domain!

$$\Longrightarrow \quad J\big|_{z_\infty}\text{- holo curve } \mathbb{C} \xrightarrow{v} M$$

conformal rescaling $z \mapsto \varepsilon_n z$

<u>Final ingredient</u> CONFORMAL INVARIANCE OF ENERGY

$v_n(s,t) = u_n(\varepsilon_n s, \varepsilon_n t)$

$$E(v) \leq E(u_n) \quad \leftarrow \underline{\text{proof}}$$

$$\begin{aligned} E(v_n) &= \int |\partial_s v_n|^2 \, ds\, dt \\ &= \int \varepsilon_n^2 \, |\partial_s u_n|^2 \cdot \frac{ds}{\varepsilon_n}\frac{dt}{\varepsilon_n} \\ &= \int |\partial_s u_n| \, ds\, dt \\ &= E(u_n) \end{aligned}$$

(compare with length in Riem. geom. being parametrisation invt)

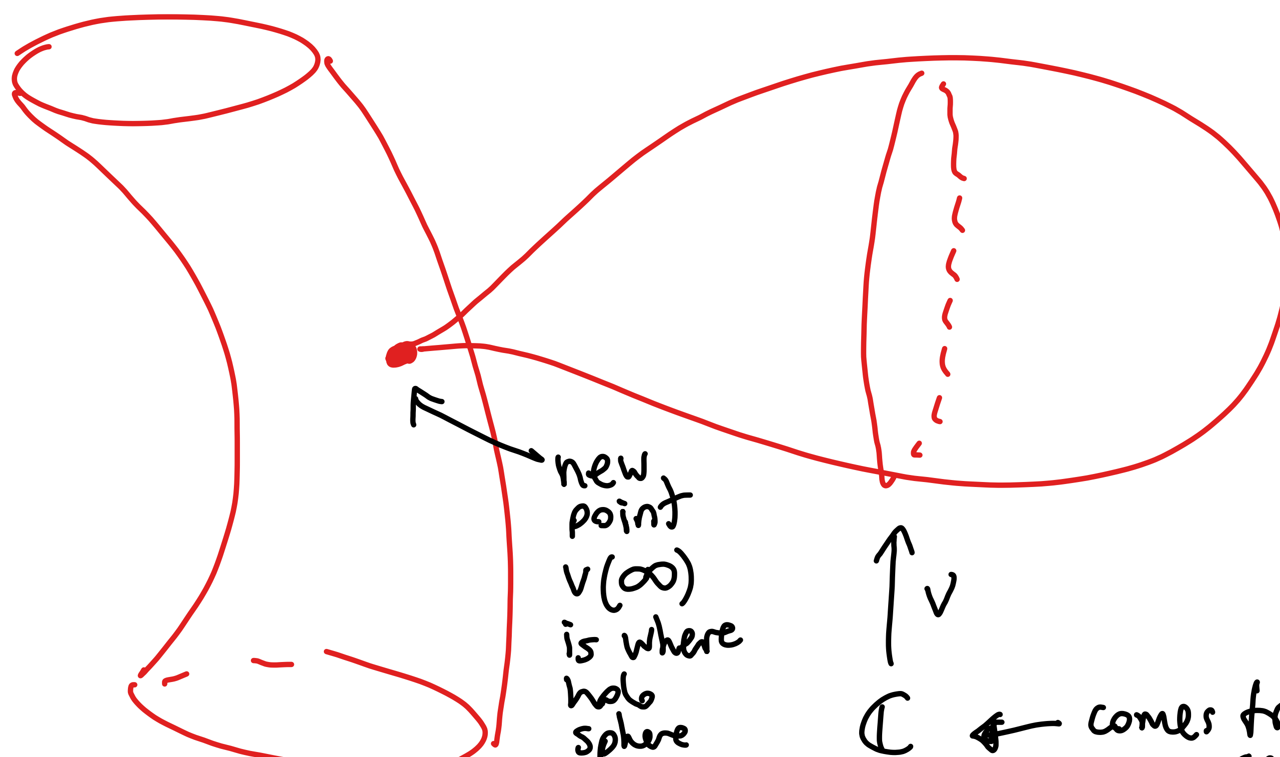
So if assume sequence u_n has bounded energy (indep. of n) then $v : \mathbb{C} \longrightarrow M$ has finite
J_{z_∞}-holo curve

Removal of singularities theorem (Gromov)

v extends holomorphically to point "∞" $\in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$

$\Longrightarrow$ $v : \mathbb{CP}^1 \longrightarrow M$ "$J|_{z_\infty}$-holo bubble"

Not totally true: see more precise comment later



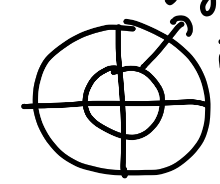
new point $v(\infty)$ is where holo sphere attaches to the MAIN COMPONENT

$u : \mathbb{R}\times S^1 \to M$

$\partial_s u + J(\partial_t u - X_H) = 0$

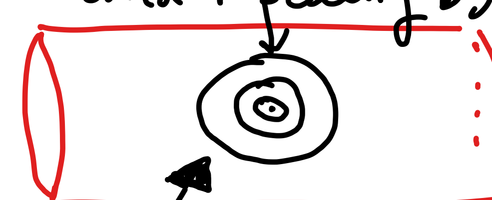
$\uparrow v$

$\mathbb{C}$ ⟵ comes from subdomains and rescaling by $1/\varepsilon_n$



larger & larger in $\mathbb{C}$

after rescale



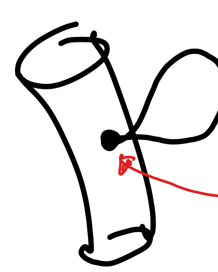
$\mathbb{R}\times S^1$

circles γ_n then $\gamma_n \to z_\infty$

$u_n|_{\gamma_n} \to u(z_\infty)$

$\parallel$

$v(\infty)$



Rmk if we're in case J = complex structure independent of domain (not just almost cx str). and M Kähler

$\Rightarrow$ could pick local cx coordinates

$v : \mathbb{C} \longrightarrow M$

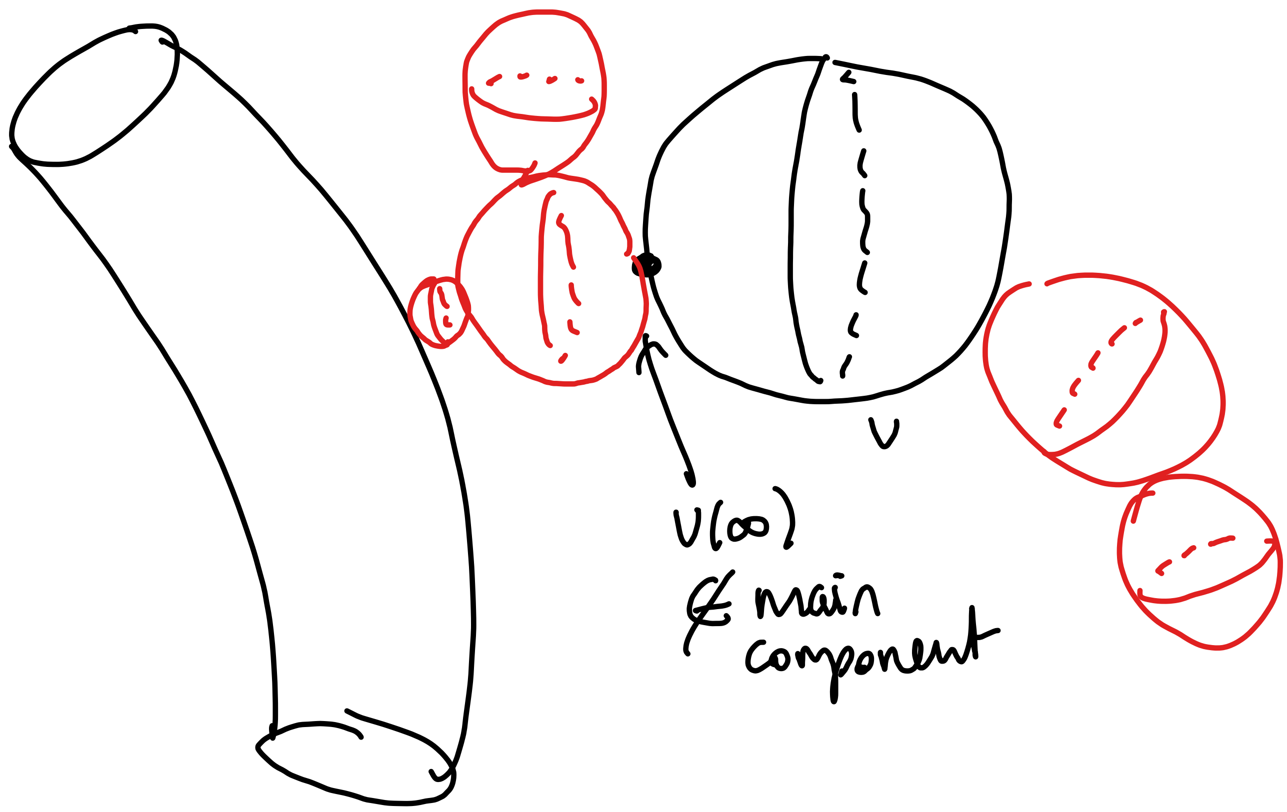
$(\text{nbhd of } \infty) \longrightarrow \text{local chart} \subseteq \mathbb{C}^n$

M compact so WLOG local chart is bounded in $\mathbb{C}^n$

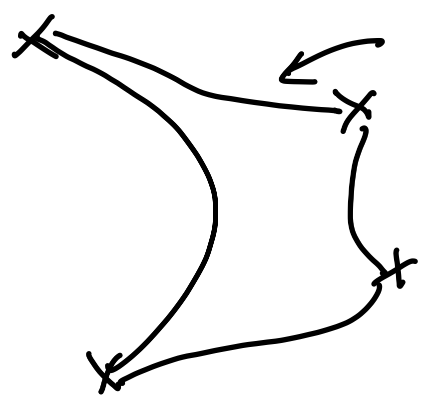
so locally just n BOUNDED holo functions defined near "∞" in $\mathbb{C}$ $\Rightarrow$ extend holo to ∞

classical removable sing. thm

Rmk Picture above is not always accurate: could have a BUBBLE TREE



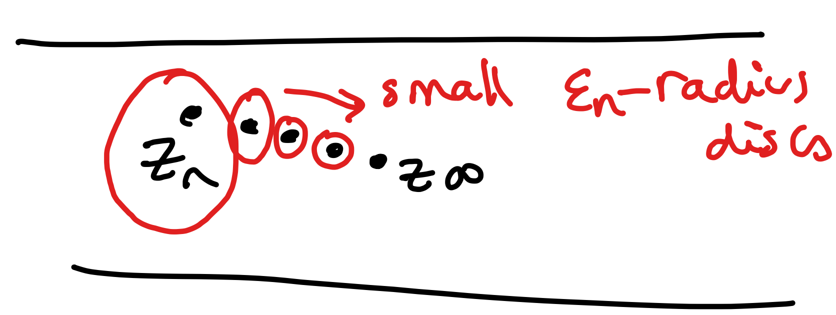
Happens for reasons similar to breaking proof in Morse case



$v_j = \lim u_n(\cdot + r_{j,n})$

various domain reparametrisations

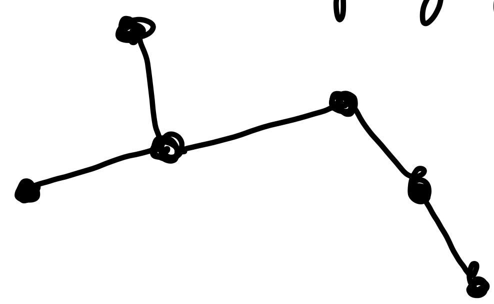
Various choices of "ε_n"-rescaling arguments & various choices of centres $z_n \to z_\infty$ giving rise to different C^∞_{loc}-limits v.



Need to look at $v(\infty)$ and $v(0)$ to see how the bubbles are joined in the tree

Rmks

1) "tree" of spheres
in sense of graph theory



no loops because "genus 0" problem
(relative to capping the ends, say)

2) Why only finitely spheres?

Usual tricks:

i) conservation of energy

Total energy of broken & bubbled "limit"
is $\lim_{n\to\infty} E(u_n) < \infty$ $\left(\begin{array}{l}\text{assume} \\ E(u_n) \leq K < \infty\end{array}\right)$

$\leftarrow$ after pass to cgt subseq.

$$\| \qquad E(\underset{u}{\text{main component}}) + \sum E(\underset{v_j}{\text{bubbles}})$$

ii) Energy quantisation

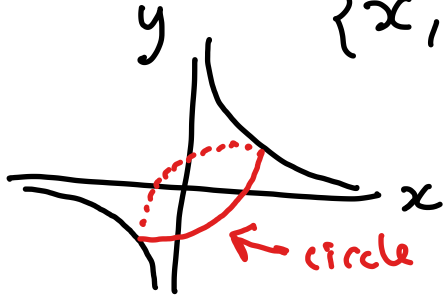
$$E(\underset{\text{sphere}}{\text{nonconstant holo}}) \geq \delta > 0.$$

so each bubbles must consume some of the total energy.

3) Only 2 components meet at each point:
Points where "irreducible components" meet are modelled on NODES so singularities arising in local cx coords as degeneration:

$$\{x, y \in \mathbb{C} : xy = t\} \overset{t\to 0}{\rightsquigarrow} \{x, y \in \mathbb{C} : xy = 0\}$$

$\leftarrow t \in \mathbb{C}$



$\underset{t\to 0}{\rightsquigarrow}$

4) Energy of holo sphere vs symplectic area

$\int |\partial_s u|^2 \, ds\, dt$ $\qquad$ $\int u^* \omega$

Same because of holo curve eqn: $\qquad$ local cx coord $z = s + it$

$$u^*\omega = \omega(\partial_s u, \partial_t u)\, ds\, dt$$
$$\underset{\partial_s u + J\partial_t u = 0}{=} \underbrace{\omega(\partial_s u, J\partial_s u)}_{= g(\partial_s u, \partial_s u) =: |\partial_s u|^2} ds\, dt$$

Crucial moment where we use $\exists$ symplectic form with compatible al cx str.

In particular: hpy invariant since ω closed

5) Bubbling is a huge problem in defining $HF^*(M)$

∂ on $CF^*(M)$ counts Floer trajectories

In proof $\partial^2 = 0$ (so chain cx)

$$\partial^2 x = \partial \sum_y \# \mathcal{M}(y,x) \cdot y$$

(y s.t. $\dim \mathcal{M}(y,x) = 0$ and transverse setup.)

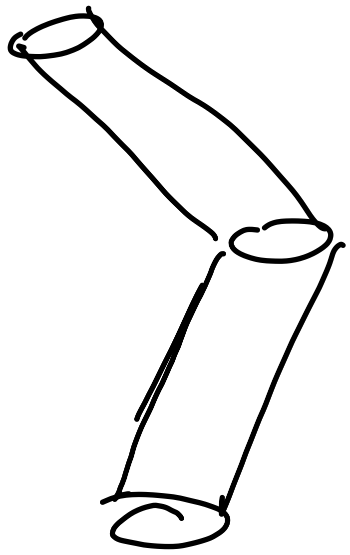
$$= \sum_{z,y} \underbrace{\# \mathcal{M}(z,y) \cdot \# \mathcal{M}(y,x)}_{} \cdot z$$

$$\underset{\text{WANT}}{=} 0$$

so want $= \#$ bdry pts of the compactification $\overline{\mathcal{M}(z,x)}$ ← 1-dim compact mfd

even so 0 mod 2 (or with orient. signs: 0)

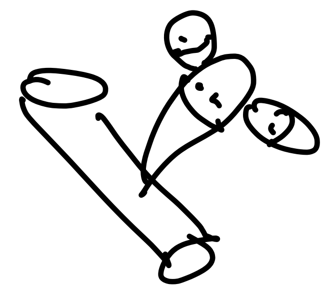
Bdry pts we need to add to $\mathcal{M}(z,x)$ to compactify



like Morse breaking

← dim arguments ⇒ can only break once

However also get limits



⇒ ruins $\partial^2=0$ proof

⇒ Floer theory literature:
Need assumptions on M to avoid this

Possible conditions

- $[\omega](\pi_2 M)=0$ "aspherical ω"

← $u: S^2 \to M$, $\int u^*\omega = 0$

⇓

Energy of holo sphere $=0$
⇒ $|\partial_s u|=0$
⇒ u constant
⇒ No holo spheres

Unfortunately: rules out almost all Kähler mfds we care about.

Examples
$c_1(M)=0$ works "Calabi-Yau case"
or
"Fano case" allowed $c_1=\lambda\cdot[\omega]$, $\lambda>0$

"semipositive" or "weak monotone"

- M closed symplectic ⇒ ω cannot be exact (by Stokes's theorem)
but in noncompact case very often see
Condition $\omega = d\theta$ exact
(⇒ aspherical by Stokes' $\int_{S^2} u^*\omega = \int_{S^2} u^* d\theta = \int_{\partial S^2=\emptyset} u^*\theta = 0$)

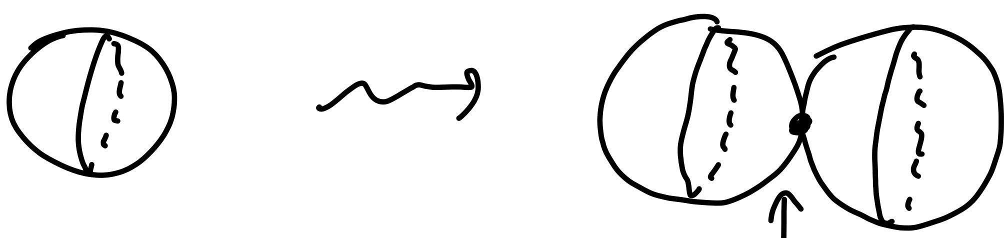
- A condition which does not rule out holo spheres in M

Trick
- Show that moduli space of holo spheres in M sweeps out (for generic J) a codimension$_{\mathbb{R}}=4$ subset in M
- show that generically 1-families of Floer traj. avoid

$\dim_{\mathbb{R}} \mathbb{R}\times S^1 = 2$
$\dim_{\mathbb{R}}$(1-family) in M $=3$
avoid codim$_{\mathbb{R}}$ 4 set

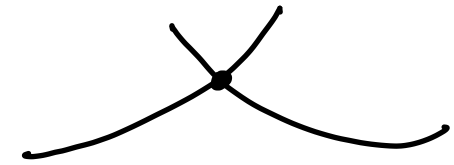
local model of all bubbling examples we saw

Example of bubbling explicitly



node

algebraic geometry picture



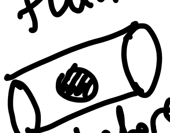
each "line" is $\mathbb{CP}^1$

secretly: conic $xy=\varepsilon$ in coords $[1:x:y]$, $(x,y)\in\mathbb{C}^2$

$$\varphi_\varepsilon : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^2$$

$$[1:z] \longmapsto \text{"}[1 : z : \tfrac{\varepsilon}{z}]\text{"} = [z : z^2 : \varepsilon]$$

$z\in\mathbb{C}$

think  from before

parameter $\varepsilon\in\mathbb{R}$

degeneration

degree 2 hol map

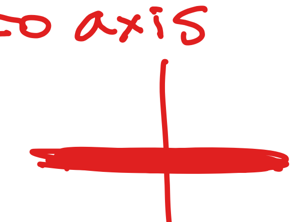
$$[z_0 : z_1] \mapsto [z_0 z_1 : z_1^2 : \varepsilon z_0^2]$$

$z = \frac{z_1}{z_0}$

$\varepsilon \to 0$:

$$\varphi_\infty : [1:z] \longmapsto [z : z^2 : 0] = [1 : z : 0]$$

in x,y coords $\mathbb{C}^2$ above: $y=0$ axis



Some topology disappeared?? now degree 1

get homology class $\varphi_{\infty *}[\mathbb{CP}^1] = 1 \cdot [\text{hyperplane class}]$

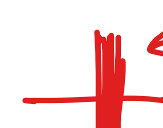
expected 2!

Why?

Reparametrisation will detect other components of "limit":

$z = \varepsilon w$
- new local cx coord w
- zooming into $w=\infty$ like in bubbling pf

$$\underset{\varepsilon w}{[1 : z]} \longmapsto [\underset{\varepsilon w}{z} : \underset{\varepsilon^2 w^2}{z^2} : \varepsilon] = [w : \underbrace{\varepsilon w^2}_{\downarrow\ 0 \text{ as } \varepsilon\to 0} : 1]$$

new limit: $w \longmapsto [w:0:1] = [1 : \underset{x}{0} : \underset{y}{\tfrac{1}{w}}]$ $\Rightarrow$ get  ← $x=0$ axis in $\mathbb{C}^2$

Meet at $[1:0:0]$ called "NODE" or "ORDINARY DOUBLE POINT"