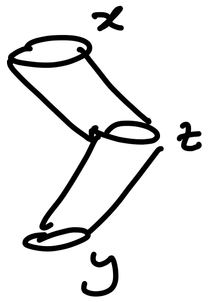
Morse 8
(Last lecture)

Continue discussion of compactness

Want $\partial^2 = 0$ so ∂ chain differential



0-dim moduli spaces

$$\partial^2 y = \sum \underbrace{\# \mathcal{M}(x,z) \cdot \# \mathcal{M}(z,y)}_{\overset{\text{want}}{\parallel}} \cdot y$$

(requires transversality so J generic)

$\# \partial \overline{\mathcal{M}(x,y)} = 0$

\# boundary points of smooth 1-mfd with boundary is even (or 0 if use orientation signs), to a smooth 1-mfd with ∂

compactification of 1-dim moduli space using broken trajectories.

once-broken for dim reasons

What we saw so far

If have sequence $u_n \in \mathcal{M}(x,y)$

$\Rightarrow$ subsequence u_n either cges or breaks

But need also converse of this:

Gluing Theorem "$\exists$ collar neighbourhood near broken trajectory"

$\overline{\mathcal{M}(x,z)}$ 1-dim'l

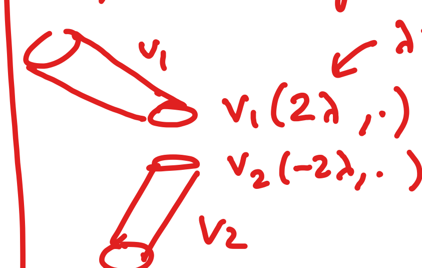
Know smooth 1-mfd for generic J (transversality thm) | Want show $\exists$ "chart" $\cong (\lambda_0, \infty] \subseteq \mathbb{R}$ | $v_1 \# v_2 \in \partial \overline{\mathcal{M}(x,z)}$

$$\exists : \mathcal{M}(x,z) \times \mathcal{M}(z,y) \times (\lambda_0, \infty] \to \overline{\mathcal{M}(x,y)}$$

$$(v_1, v_2, \lambda) \longmapsto v_1 \#_\lambda v_2$$

$$(v_1, v_2, \infty) \longmapsto v_1 \# v_2 \quad \text{broken solution}$$

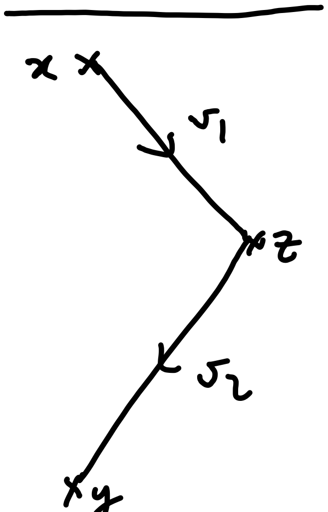
idea:
before breaking



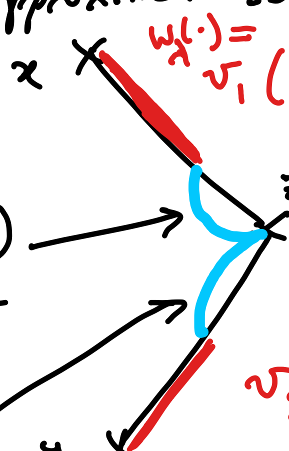
gluing procedure

so 1-free parameter λ involved in gluing for each breaking

Morse case



produce an approximate solution w_λ



$w_\lambda(\cdot) = v_1(\cdot + 2\lambda)$ for $s \leq -\lambda$

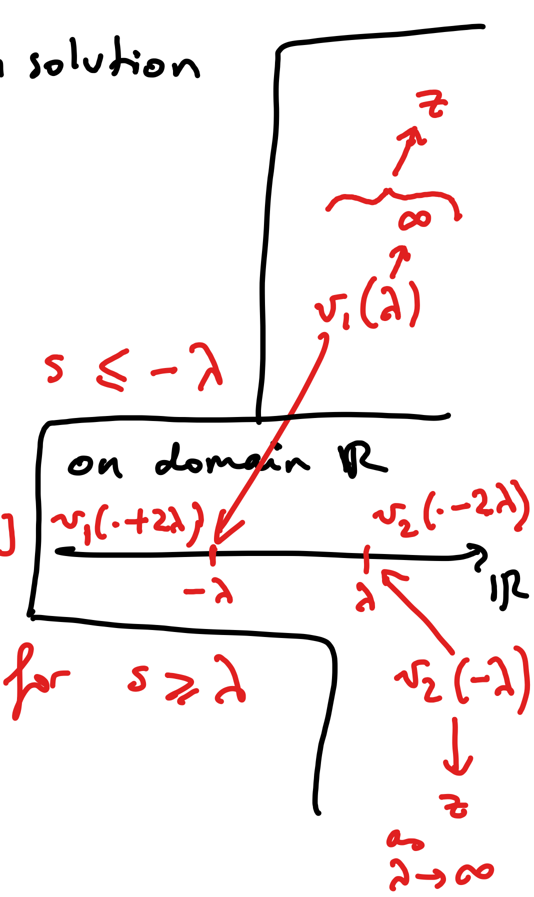
$w_\lambda(s) = z$ for $s \in [-\lambda+1, \lambda-1]$

$w_\lambda(\cdot) = v_2(\cdot - 2\lambda)$ for $s \geq \lambda$

use $\exp_z(\cdot)$ to interpolate between $v_1(\lambda)$ and z

$s \in [-\lambda, -\lambda+1]$

use $\exp_z(\cdot)$ go from z to $v_2(-\lambda)$ for $s \in [\lambda-1, \lambda]$



$$F(u) = \partial_s u - \nabla f\big|_u$$

$$F(w_\lambda) = \begin{cases} F(v_1(\cdot + 2\lambda)) = 0 & s \leq -\lambda \text{ since } v_1 \text{ Morse traj} \\ F(v_2(\cdot - 2\lambda)) = 0 & s \geq \lambda \text{ as } v_2 \text{ traj} \\ F(z) = 0 & s \in [-\lambda+1, \lambda-1] \\ F(\exp_z(\dots)) & s \in \text{remaining two intervals of length } 1 \end{cases}$$

$$F(\exp_z(\dots)) = \underbrace{\partial_s \exp_z(\dots)}_{\approx 0} - \underbrace{\nabla f\big|_{\exp_z(\dots)}}_{\approx \nabla f|_z = 0}$$

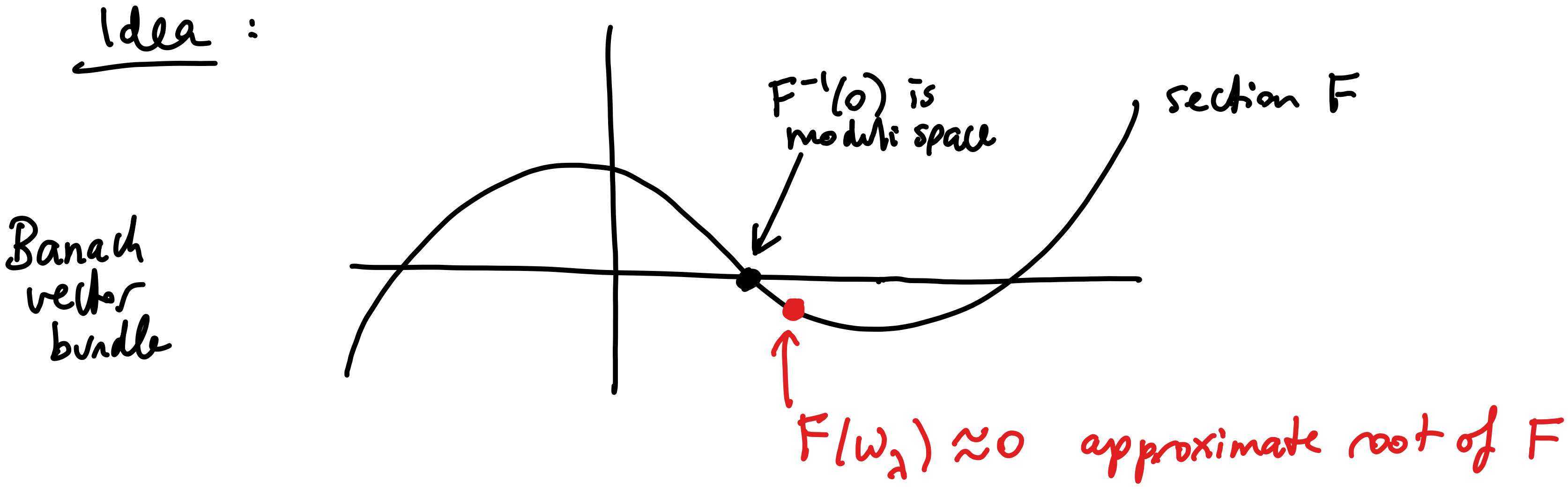
(≈ 0 since very close to z (exponential decay of v_1, v_2 near z))

$\Rightarrow$ $F(w_\lambda) = 0$ except small on two small intervals.

Gluing thm takes a lot of time to prove, even in my Cambridge Part III notes I just give an overview sketch.

$\rightsquigarrow$ Details: book by Matthias Schwarz, Morse Homology

Idea:

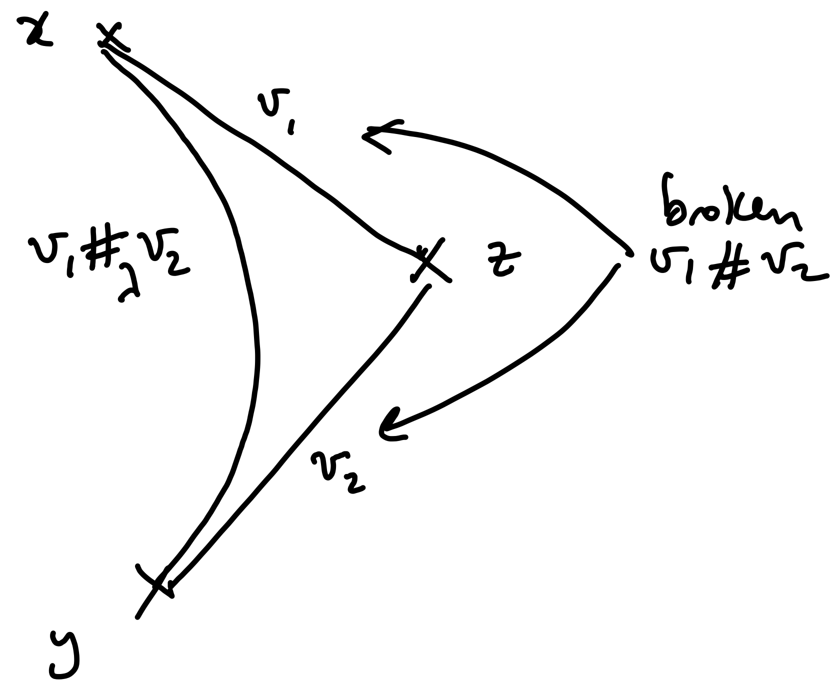


Run Newton iteration method (or Picard method) to find honest solution near w_λ.

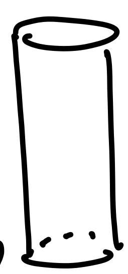
Basically: implicit function thm argument (Banach fixed pt thm)

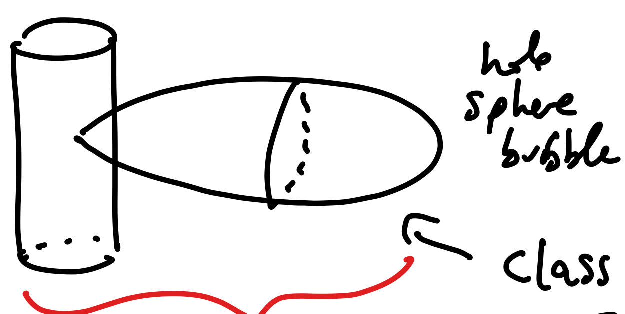
One builds a "unique" family of honest solutions $v_1 \#_\lambda v_2$ associated to w_λ for $\lambda \geq \lambda_0$ (LARGE)

(explained in Part III)



Floer case excluding bubbling





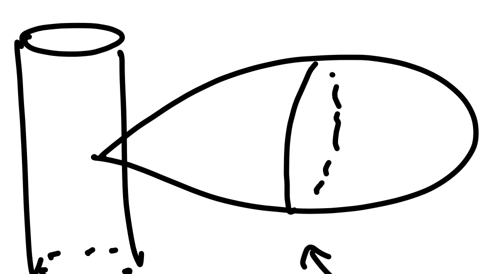
class $A \in H_2(M)$

$\pi_2 M \longrightarrow H_2 M$
Hurewicz

total index $= 2$

Fredholm operator index is additive ← (dim Ker − dim Coker)

index 2
since 1-family of traj
and have $\mathbb{R}$-transl^n reparam.



index ≥ 3

$\Rightarrow$ index $= 2 -$ this ≤ -1

so this moduli space has expected dimension < 0
(or virtual dimension)

means: dim when J generic so transversality holds

$\Rightarrow \varnothing$ moduli space for generic J
$\Rightarrow$ does not happen

Fact • virtual dim $\Big(\overbrace{J\text{-hol spheres in class } A \in H_2(M)}^{\mathcal{M}_A}\Big)$

$$= \underbrace{2n}_{\dim_{\mathbb{R}} M} + 2c_1(A)$$

$c_1(A)$ ← evaluate $H^2(M,\mathbb{Z})$ with $H_2(M,\mathbb{Z})$

$c_1(TM, J)$ ← cx vector bdle using J

• reparametrisation freedom of domain of J-hol sphere:

$$\mathrm{Aut}(\mathbb{CP}^1) = PSL(2,\mathbb{C}) \quad \leftarrow \dim_{\mathbb{R}} = 6$$

(Möbius maps)

$v \circ \varphi^{-1} : \mathbb{CP}^1 \to M$ has same image

$$\mathcal{M}_A \times \mathbb{CP}^1 \xrightarrow{ev} M$$

what $\mathcal{M}_A$ sweeps out in M

$$\big((v : \mathbb{CP}^1 \to M),\ p\big) \longmapsto v(p)$$

$v_*[\mathbb{CP}^1] = A$

↺ action $\mathrm{Aut}(\mathbb{CP}^1)$: $(v,p) \longmapsto (v\circ\varphi^{-1}, \varphi\circ p)$

$ev \searrow \ \swarrow ev$ $v(p)$

How much of M is "covered":

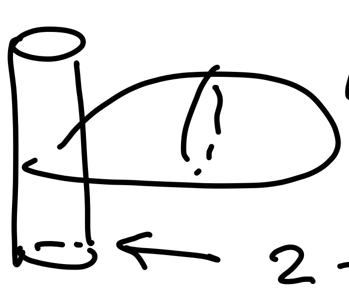
$$\text{"dim"(image)} = \underbrace{2n + 2\alpha}_{\dim \mathcal{M}_A} \ \underbrace{-\ 6}_{\mathrm{Aut}\,\mathbb{CP}^1} \ \underbrace{+\ 2}_{\substack{p \in \mathbb{CP}^1 \\ 2 \dim_{\mathbb{R}} \text{ freedom}}}$$

$\alpha = c_1(A)$

$$= 2n - 4 + 2\alpha$$

Really problematic case

negative Chern classes very dangerous



$\alpha = -1$ (or $\alpha \ll 0$ even worse) ← e.g. compose v with branched covers $\mathbb{CP}^1 \to \mathbb{CP}^1$ to make $c_1 \ll 0$ spheres

$2 - \alpha = 3$ (or $\gg 0$) ← expect these exist! since positive dim

⇝ Fukaya – Oh – Ohta – Ono developed methods called Kuranishi structures

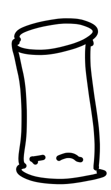
Possible conditions on M to avoid bubbling

① ω aspherical : $\omega(\pi_2 M) = 0$ $\Rightarrow$ no nonconst J-hol spheres

② Calabi-Yau case : $c_1 = 0$
or just ask $c_1(\pi_2 M) = 0$

③ Fano case or "Monotone" case:
$$c_1 = \underset{k>0}{k}\,[\omega] \in H^2(M, \mathbb{R})$$

② "dim" image $= 2n-4$
so spheres generically live in $\mathrm{codim}_{\mathbb{R}} = 4$ subset


$\leftarrow$ 2 dim domain

1-family in proof of $\partial^2 = 0$

live in $2+1 = 3$ $\dim_{\mathbb{R}}$'l subset

Just show : generic J $\Rightarrow$ these are disjoint subsets
$\Rightarrow$ no bubbling in $\partial^2 = 0$ proof

③ $\alpha = c_1(A) = k\, \underbrace{\omega(A)}_{>0}$

>0 since energy of J-hol sphere
non constant say.

$\Rightarrow$ $c_1(A) \geq 1$ since $c_1 \in H^2(M, \mathbb{Z})$
$A \in H_2(M, \mathbb{Z})$

$\Rightarrow$ $2\alpha \geq 2$

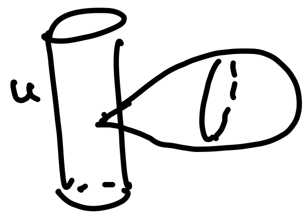
$\Rightarrow$ "dim" (image) $= 2n - 4 + 2\alpha$

index ≥ 3 case, so $2\alpha \geq 3$, we already discussed (no issue)

left with : $2\alpha = 2$

$\Rightarrow$ "dim" (image) $= 2n - 4 + 2 = 2n - 2$

$2n-2$



index $2\alpha = 2$

index $= (2 - \text{that}) = 0$ means **no** $\mathbb{R}$-reparam freedom, so must have

$$u(\cdot + r, \cdot) = u(\cdot, \cdot) \quad \forall r \in \mathbb{R}$$

$\Rightarrow$ u is s-independent

$\Rightarrow$ $u = x = y$ is s-indep. Ham 1-orbit

But we have $x \neq y$ ($\dim \mathcal{M}(x,y) = 1$)

$\Rightarrow$ does not happen

(Rmk. or, say "dim" (image) $= 2n-2$ $\quad \mathrm{codim}_{\mathbb{R}} = 2$

so spheres avoid (generically) the Ham 1-orbits.

$\longrightarrow$ 1-dim subspace)

How we count $\# \mathcal{M}(x,y)$

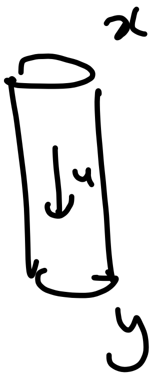
$$\partial : FC^*(H,J) \longrightarrow FC^{*+1}(H,J)$$

$$\parallel$$

$$\bigoplus_{y \text{ Ham 1-orbit for } H} \mathbb{K}\, y$$

$\mathbb{K}$ = Novikov field ← various versions

$= \{ \sum a_i t^{r_i} : a_i \in \text{ground field } \mathbb{F},\ r_i \to \infty \text{ in } \mathbb{R} \}$



$$\partial y = \sum_{\substack{u \in \mathcal{M}(x,y) \\ \dim \mathcal{M}(x,y) = 1}} \pm\, t^{E(u)} \cdot x$$

$E(u)$ ← energy $\geq 0 \in \mathbb{R}$

J generic so transverse setup

orientation sign (if char field $\neq 2$)

formal parameter t

Why?

$\mathcal{M}(x,y)$ could be ∞ set

if put bound on energy $E(u) \leq K$ then we showed compactness

But could have u_n with $E(u_n) \to \infty$

So doing like a formal completion in t to ensure convergence

Invariance : Floer's continuation method

Morse $\quad f_s, g_s \quad s \in \mathbb{R}$ $\qquad f_s = \begin{cases} f_- & s \ll 0 \\ f_+ & s \gg 0 \end{cases}$

i.e. $\exists\, s_- \in \mathbb{R}$ & this holds for $s \leq s_-$

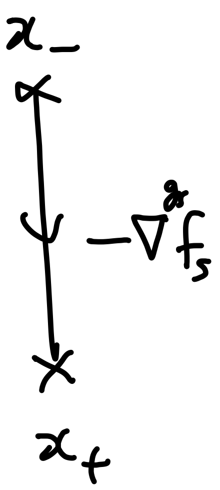
Think s very negative

$f_s : M \to \mathbb{R}$ smooth functions need not be Morse $\qquad f_\pm : M \to \mathbb{R}$ Morse

g_s Riemannian metrics

$\Rightarrow \nabla^{g_s} f_s$ vector fields

$g_s(\nabla^{g} f_s, \cdot) = df_s$



$$MC^*(f_+, g_+) \longrightarrow MC^*(f_-, g_-)$$

$$x_+ \longmapsto \sum_{\dim \mathcal{M}=0} \# \mathcal{M}(x_-, x_+) \cdot x_-$$

Morse continuation solutions $\longrightarrow$ $u : \mathbb{R} \to M$

$$\partial_s u = -\nabla^{g_s} f_s$$

$$\left(= \begin{cases} -\nabla f_- & s \ll 0 \\ -\nabla f_+ & s \gg 0 \end{cases} \right.$$

flows used to define MC^* groups

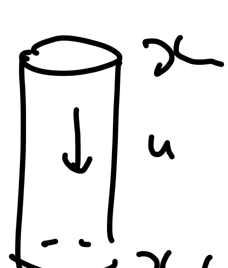
Important Rmk

No $\mathbb{R}$-reparametrisation freedom

$\mathcal{M}(x_-, x_+) = W(x_-, x_+)$

always parametrised maps now

Floer case $\quad \omega$ symplectic form



$J_s \quad \omega$-compatible almost complex structures

$$J_s = \begin{cases} J_- & s \ll 0 \\ J_+ & s \gg 0 \end{cases}$$

$H_s : M \to \mathbb{R}$

$$H_s = \begin{cases} H_- & s \ll 0 \\ H_+ & s \gg 0 \end{cases}$$

$$\varphi_{-+} : FC^*(H_+, J_+) \longrightarrow FC^*(H_-, J_-)$$

$$x_+ \longmapsto \sum \#\begin{pmatrix}\text{Floer continuation} \\ \text{solutions}\end{pmatrix} \cdot x_-$$

$u : \mathbb{R} \to \mathcal{L}M = C^\infty(S^1, M)$

$\partial_s u = -\nabla^{g_s} A_{H_s}$

$$\partial_s u + J_s(\partial_t u - X_{H_s}) = 0$$

exercise $\quad J_s X_{H_s} = -\nabla^{g_s} H_s$

$$\left(\omega(\cdot, X_{H_s}) = dH_s\right)$$

$$A_{H_s}(\underset{\mathcal{L}M}{\underset{\in}{x}}) = -\int \underset{\text{cap}}{\bar{x}^*}\omega + \int H_s(x)\, ds\, dt$$

cap $\bar{x}$, x

for contractible loops

(non-contractible loops: pick representative loop in free hpy class $S^1 \to M$ of x)

Then "cap" is really cylinder

Rmk For transversality:
generic f_s is enough
or
generic g_s is enough
or both.

For continuation solution, count is more precisely

$$\partial x_+ = \sum \pm\, t^{\underbrace{A_{H_-}(x_-) - A_{H_+}(x_+)}} \cdot x_-$$

hpy invt relative to ends $x_\pm$

not the energy (not hpy energy)

Correct analogue of $f_-(x_-) - f_+(x_+)$ of Morse theory case $= \int \partial f_s|_u \, ds$

Energy

$$E(u) = \int |\partial_s u|^2_{g_s} \, ds\, dt$$

$$= \int \omega(\partial_s u, \underbrace{J_s \partial_s u}_{= \partial_t u - X_{H_s}}) \, ds\, dt$$

$$= \int u^* \omega - \int \underbrace{dH_s(\partial_s u)}\, ds\, dt$$

$$\partial_s (H_s \circ u) = dH_s \circ \partial_s u + \partial_s H_s \big|_u$$

NEW TERM

$$= \int u^*\omega - \int \partial_s (H_s \circ u) + \int \partial_s H_s$$

$$= \underbrace{A_{H_-}(x_-) - A_{H_+}(x_+)}_{\text{htpy invt rel. ends}} + \underbrace{\int \partial_s H_s \big|_u \, ds\, dt}_{\text{not htpy invt}}$$

H_s s-independent for $s \in \mathbb{R} \setminus [s_-, s_+]$

$s_- \ll 0$, $s_+ \gg 0$

$$\left| \int \partial_s H_s \right| \le (s_+ - s_-) \max_M |H_s|$$

bounded a priori

$\Rightarrow \quad E(u) \le$ difference of action $+$ a priori bound

$\Rightarrow \quad \pm t \overbrace{}^{\text{difference of action}} \quad x_-$

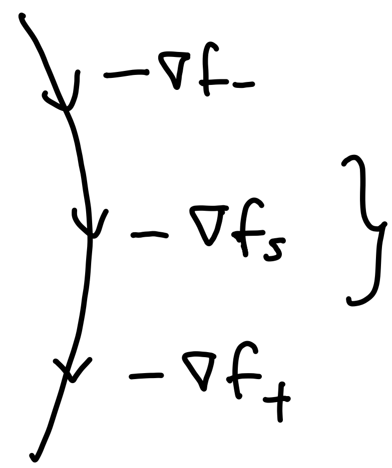
will go to ∞ if $E(u_n) \to \infty$ so all ok.

Is φ_{-+} a chain map?

What breaking?

Morse case

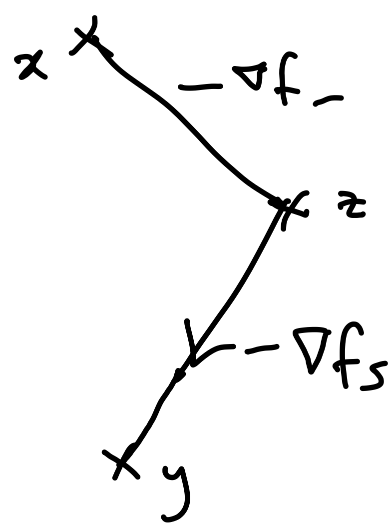
1-family



} Key s-dependence is on a compact subset of the domain $\mathbb{R}$
$\Rightarrow$ Arzela-Ascoli ensures C^∞-cgce
$\therefore$ no breaking!

$\Rightarrow$ only breaking is near ends
so breaking analysis is same as for MC^* case
(That's why crucial that we use same PDE setup near the ends)

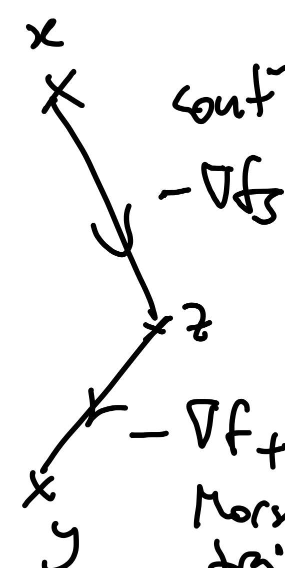
$\Rightarrow$



Morse traj for f_-

Continuation soln

or:



cont^n $-\nabla f_s$

Morse traj for f_+

counted by

$\partial_- \circ \varphi_{-+}(y)$

$\uparrow$

∂ for $MC^*(f_-, g_-)$

counted by

$\varphi_{-+} \circ \partial_+(y)$

$\#\partial \underbrace{\overline{\mathcal{M}(x,y)}}_{\text{compactified smooth 1-mfd with } \partial} = 0 \Longrightarrow$

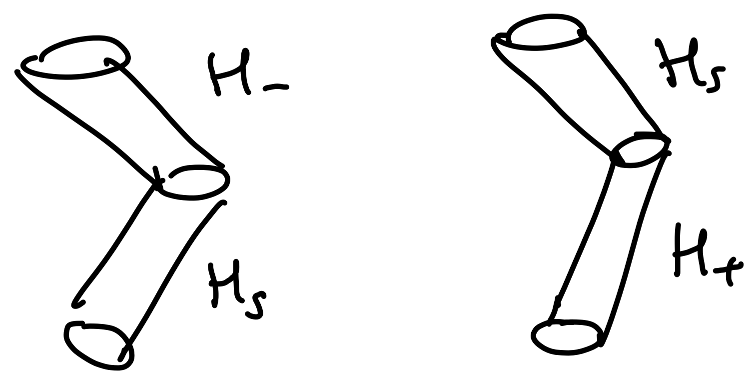
$$\partial_- \circ \varphi_{-+} + \varphi_{-+} \circ \partial_+ = 0$$

says φ_{-+} is chain map

$$\Longrightarrow \quad [\varphi_{-+}] : MH_*(f_+, g_+) \to MH_*(f_-, g_-)$$

<u>Floer case</u> similar

(for transversality either perturb H_s
or " J_s
or both)

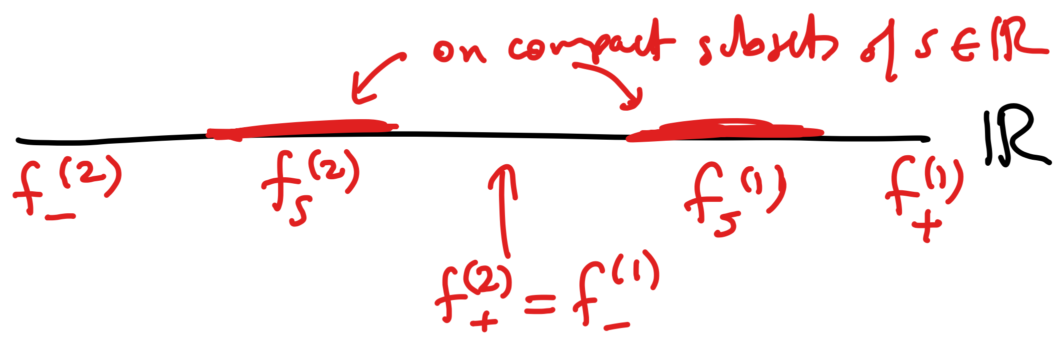


<u>Thm</u> ← (crucial M closed)

1) $[\varphi_{-+}]$ independent of choice of (generic) f_s, g_s
resp. H_s, J_s Floer case

2) compose well:

$$[\varphi_{f_s^{(2)}}] \circ [\varphi_{f_s^{(1)}}] = [\varphi_{f_s^{(2)} \circ f_s^{(1)}}]$$



3) $[\varphi_{-+}] = id$ if $f_s = f$ Morse, $g_s = g$ $\qquad H_s = H$, $J_s = J$
(f, g) generic

4) $[\varphi_{-+}]$ isomorphism
($\Longrightarrow$ INVARIANCE OF MH^* AND FH^*
So independent of f, g resp. H, J)

proof ideas

1) 2 sets of choices $\Longrightarrow$ 2 chain maps $\varphi^{(0)}, \varphi^{(1)}$

interpolate choices : $f_s^{(\lambda)}, g_s^{(\lambda)}$ $\qquad \lambda \in [0,1]$

PARAMETRISED MODULI SPACE

$$\bigcup_{\lambda \in [0,1]} \mathcal{M}(x, y; f_s^{(\lambda)}, g_s^{(\lambda)})$$

← union has dim = 1 : new λ param

← call this $\mathcal{M}_\lambda$

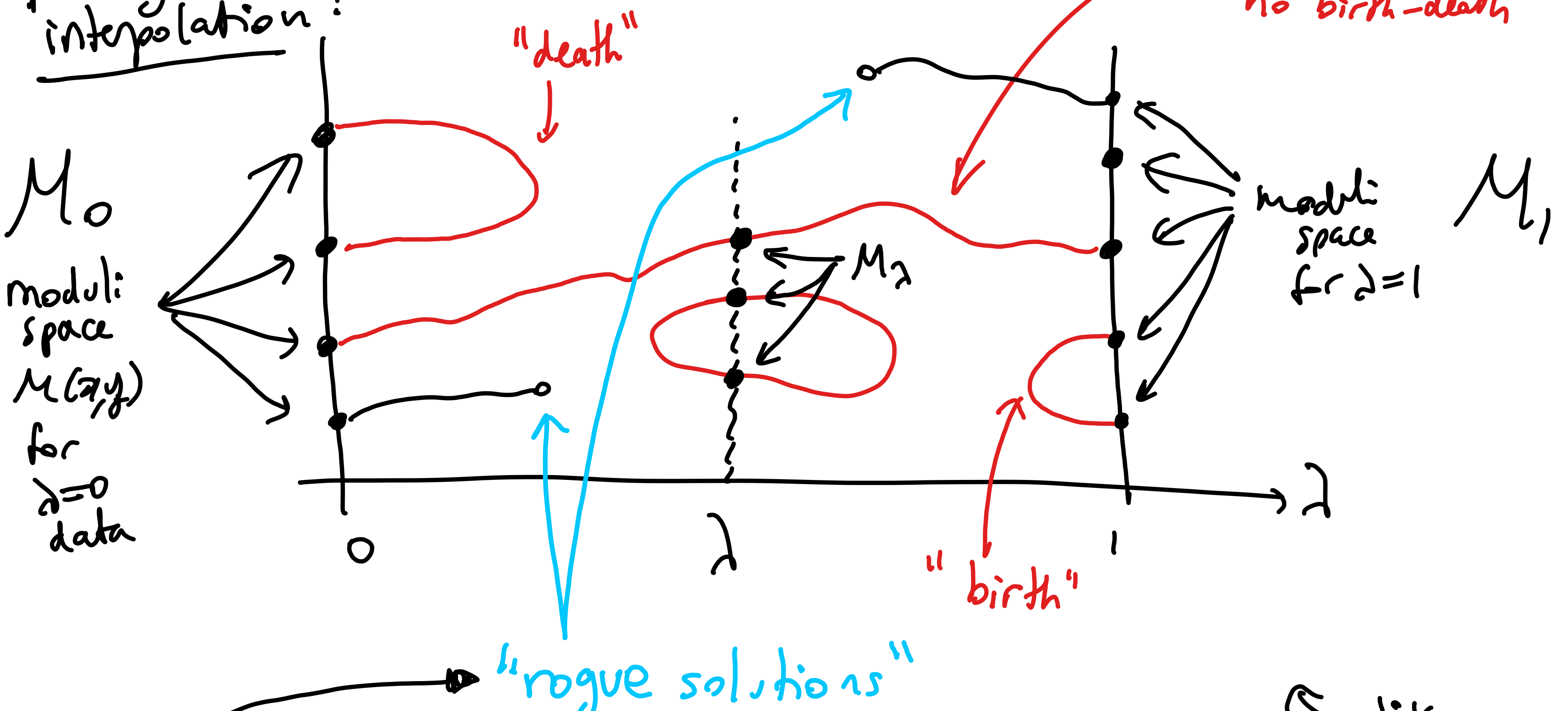
each $\mathcal{M}_\lambda$ has dim = 0 (φ_{-+} counts index 0 sol/ns)

so pairs : (u, λ), $\partial_s u = -\nabla^{g_s^{(\lambda)}} f_s^{(\lambda)}$, $\lambda \in [0,1]$

so get nicest possible birth-death setup

nice case: 1-cobordism no birth-death

For generic interpolation :



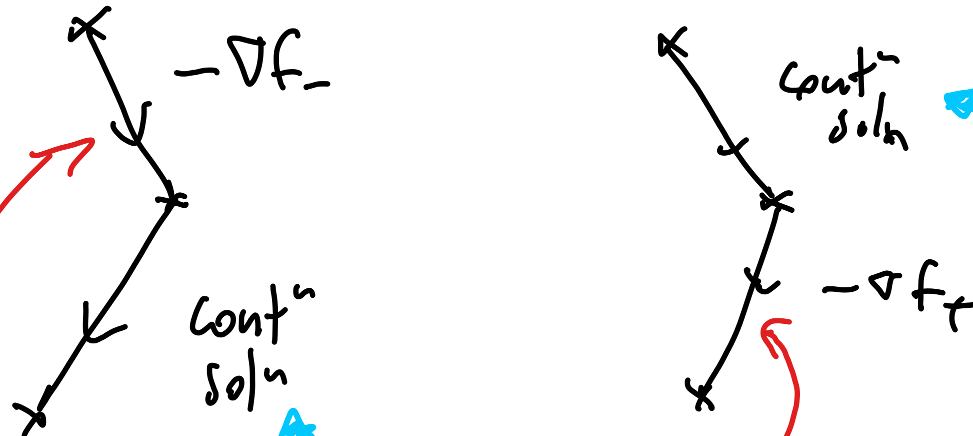
← like "bifurcation analysis" in classical ODE/PDE theory

birth-death does not affect the counts of $\mathcal{M}_\lambda$ mod 2 (or in fact over $\mathbb{Z}$ if use orientation signs)

happens if breaking . . . .

total index $\mu_\lambda = 0$ (for fixed λ)

... breaking



if nonconst
$\Rightarrow \exists$ $\mathbb{R}$-reparam
$\Rightarrow$ index ≥ 1

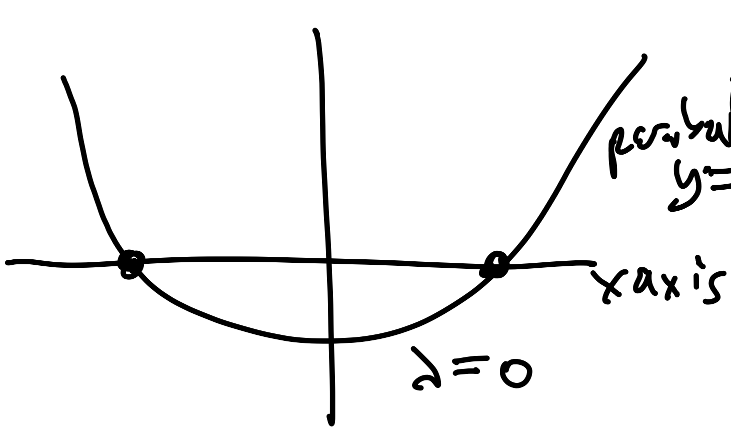
$\Rightarrow$ index ≤ -1
opposite of since total index $= 0$

How possible these moduli spaces of cont^n sol^{n}s are not empty?

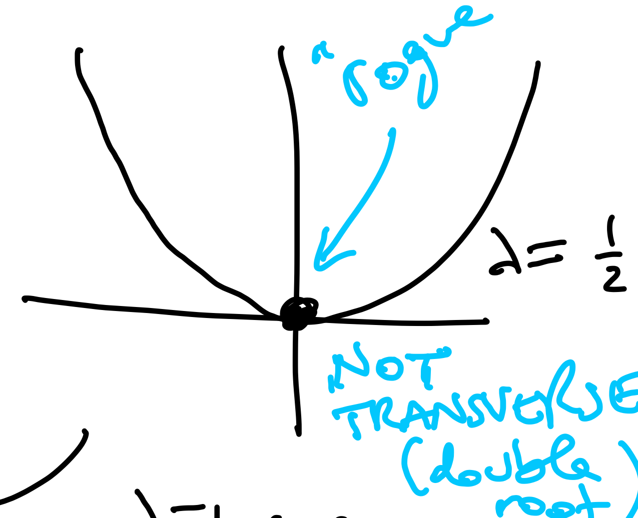
Reason: f_λ, g_λ (resp H_λ, J_λ)
not generic for that particular value of λ

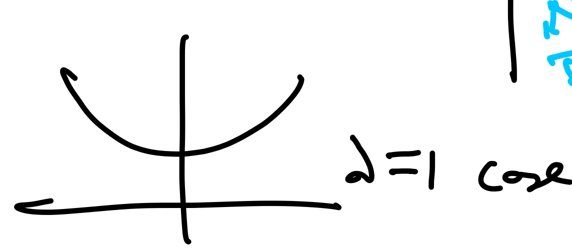
Philosophy: can make f, g generic
but not in 1-family!

Compare: when is parabola transverse to x-axis?



for all other λ
it is transverse





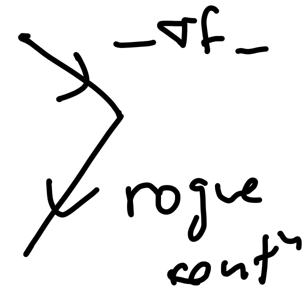
upshot

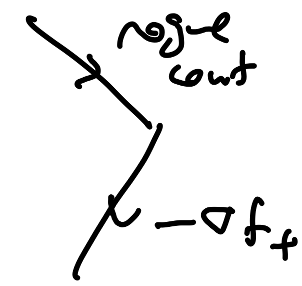
$$0 = \# \, \partial \Big(\underbrace{\overline{\bigcup_{\lambda \in [0,1]} \mathcal{M}_\lambda}}_{\text{compact 1-mfd with } \partial} \Big)$$

compactify using broken solns involving rogue solns

$$\Downarrow$$

$$\varphi^{(0)} - \varphi^{(1)} = \partial_- \circ K + K \circ \partial_+$$



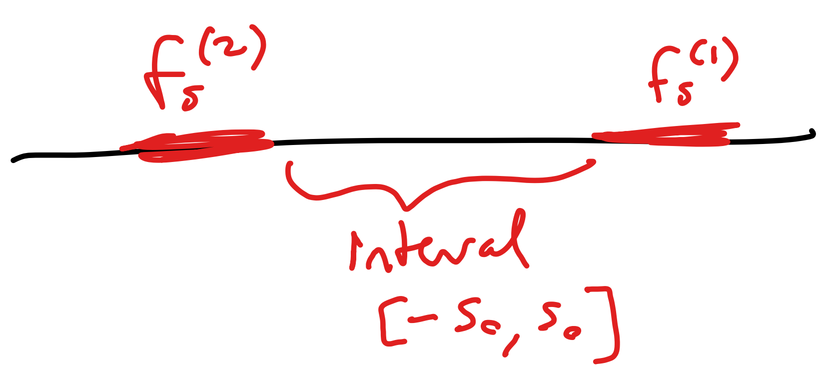


K called chain homotopy

$$\Longrightarrow \quad [\varphi^{(0)}] = [\varphi^{(1)}] \qquad \checkmark$$

2) <u>Compose well</u>

One shows at chain level that if $s_0 \gg 0$



need to look at Fredholm operators by hand

the set bijection of moduli spaces, so an iso at chain level!

So use (1) to get into this situation.

3) $\varphi = \mathrm{id}$ if constant data

f, g generic (transversality holds) so that have transversality for all Morse solutions $\partial_s u = -\nabla f$

$\Longrightarrow$ dim {const$^{\text{t}}$ solns) never 0 due to $\underline{\mathbb{R}\text{-reparam. freedom}}$

$u(\cdot + r, \cdot)$

UNLESS s-independent

$\Longrightarrow$ u = s-independent Ham 1-orbit

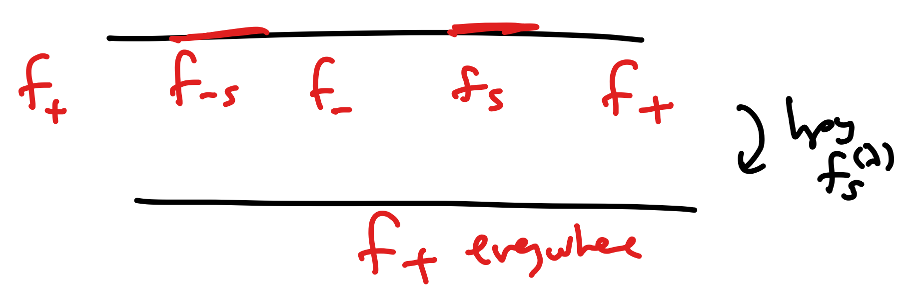
$\Longrightarrow$ $\varphi_{-+}(x) = x$ is identity.

data is s-independent so $\underline{\text{can}}$ translate s!

4) $[\varphi_{f_s}]^{-1}$?

$$\underbrace{[\varphi_{f_{-s}}]}_{\text{reverse hpy}} \circ [\varphi_{f_s}] \underset{(2)}{=} [\varphi_{f_{-s} \# f_s}] \underset{(1)}{=} [\varphi_{f_+}] \underset{(3)}{=} \mathrm{id}$$

f_+ s-independent

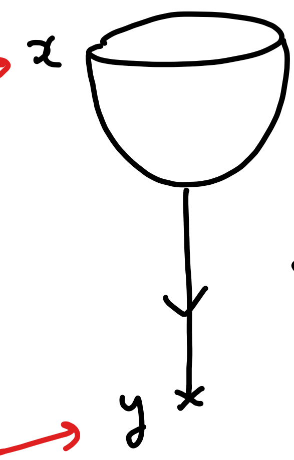


$\Longrightarrow [\varphi_{f_s}]^{-1} = [\varphi_{f_{-s}}]$. ∎

Piunikhin–Salamon–Schwarz
"PSS map"

Morse $\longrightarrow$ Floer map

$$c^*: MH^*(f) \longrightarrow FH^*(H)$$
$$y \longmapsto \sum \#(\text{spiked discs}) \cdot x$$



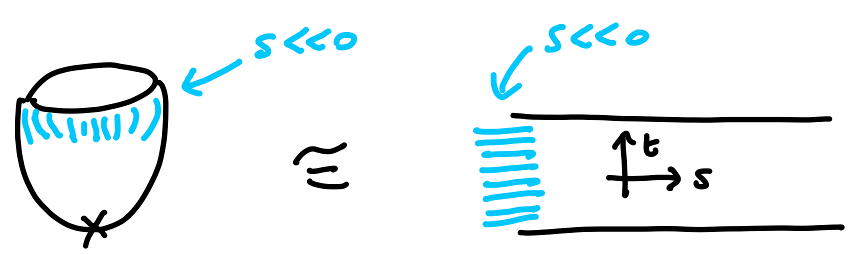
1-periodic Ham orbit for H

$u: \mathbb{C} \to M$ see below

$v: [0,\infty) \to M$
$\partial_s v = -\nabla f$

$y \in \mathrm{crit}(f)$

description of u



$\mathbb{C}\setminus\{0\} \cong \mathbb{R}\times S^1$

$e^{-2\pi(s+it)} \longleftrightarrow (s,t)$

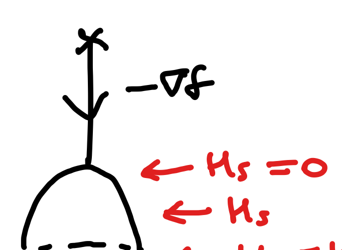
$$\begin{tikzcd} \mathbb{R}\times S^1 \arrow[r] \arrow[rr, bend right, "u(s,t)"'] & \mathbb{C}\setminus\{0\} \arrow[r] & M \end{tikzcd}$$

$$\partial_s u + J(\partial_t u - X_{H_s}) = 0$$

$$H_s = \begin{cases} H & \text{for } s \ll 0 \\ 0 & \text{for } s \gg 0 \end{cases} \quad \longleftarrow \text{so near } 0 \in \mathbb{C} \text{ get } J\text{-hol curve equation } \partial_s u + J\partial_t u = 0$$

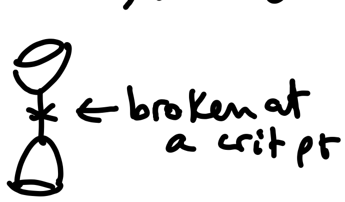
Thm c^* is an isomorphism

proof idea define "inverse" map γ^* counts

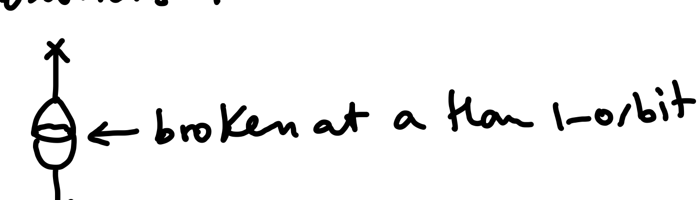


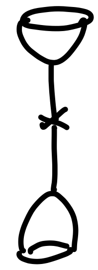
$c^*\circ\gamma^*, \gamma^*\circ c^*$ count broken solutions:

$c^*\circ\gamma^*$

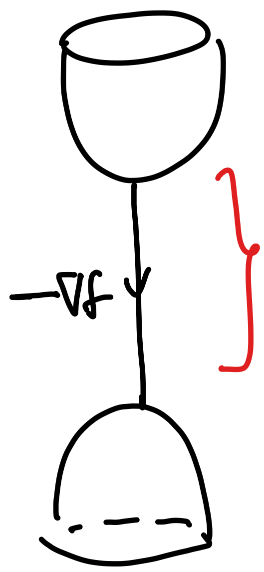


$\gamma^*\circ c^*$



How to prove  count is identity on cohomology?

Trick: "Interpolating moduli space" trick



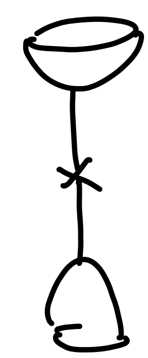
free parameter $\ell > 0$

$v : [0,\ell] \to M$

$\partial_s v = -\nabla f$

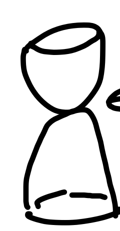
Breaking possibilities:

$\ell \to \infty$



counted by $c^* \circ \gamma^*$

$\ell \to 0$

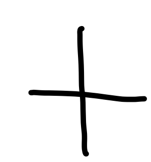


node, like in the discussion of sphere bubbling:

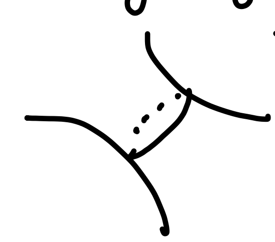
Local model

$xy = 0$

$x, y \in \mathbb{C}$



admits gluing

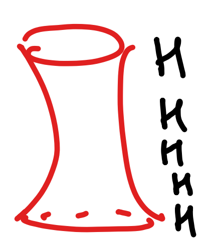


$xy = t \neq 0 \in \mathbb{C}^*$

so after gluing argument get

$\longrightarrow$ can hpe data $\lambda \in [0,1]$

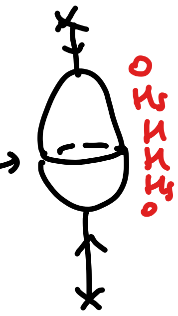
(· parametrised moduli space argument
· count rogue solns, get chain hpy)



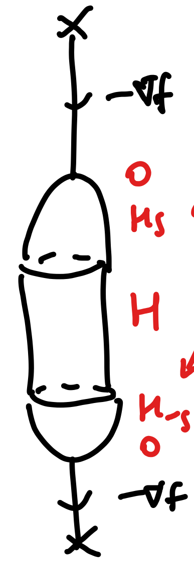
$\Longrightarrow$ up to chain hpy, counting continuation cylinders with $H_s = H$ s-indep. so identity map on cohomology

$\therefore \quad c^* \circ \gamma^* = \mathrm{id}.$

proof of $\gamma^* \circ c^* = \mathrm{id}$:



Interpolating moduli space :

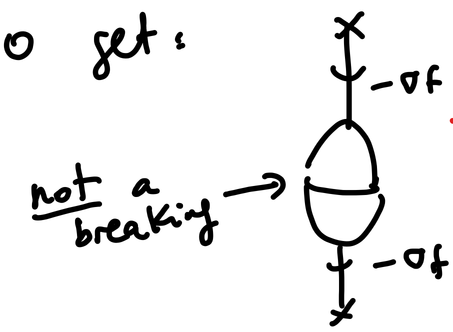


can hpe $H_s \to 0$ as $\ell \to 0$

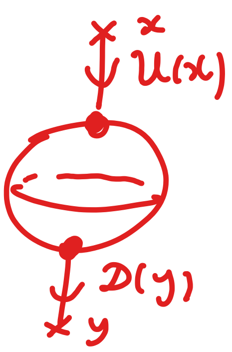
cylinder $[-\ell,\ell]\times S^1 \to M$

free parameter ℓ

Breaking : $\begin{cases} \ell \to +\infty \text{ get} \\ \ell \to 0 \text{ get:} \end{cases}$



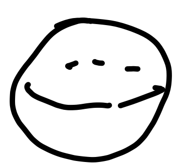
} $H_s = 0$ everywhere, so holo sphere



Finally, need a dimension argument

moduli space of spheres with 2 marked points evaluating to $\mathcal{U}(x), \mathcal{D}(y)$:

$$\mathcal{M}_A \left\{ \text{sphere} \times \mathbb{CP}^1 \times \mathbb{CP}^1 \right\} \Big/ \mathrm{Aut}(\mathbb{CP}^1) \xrightarrow[ev]{ev} M \times M$$



$$M\times M \supseteq \underbrace{\mathcal{U}(x)}_{\parallel} \times \underbrace{\mathcal{D}(y)}_{\parallel}$$

$\mathcal{U}(x) = \{ p \in M : -\nabla f \text{ flow } \underline{\text{back}} \text{ in time} \to x \}$

$\mathcal{D}(y) = \{ p \in M : -\nabla f \text{ flow } \underline{\text{forward}} \text{ in time} \to y \}$

reparametrisation:

(u, p_1, p_2)

$\downarrow \varphi \in \mathrm{Aut}\,\mathbb{CP}^1$

$(u\circ\varphi^{-1}, \varphi p_1, \varphi p_2)$

ev is invt under reparam.

$$\text{"dim"} = 2n + 2c_1(A) + 2 + 2 - 6 - |x| - (2n - |y|) = 2c_1(A) + |y| - |x| - 2$$

if one carefully checks the dimensions involved in the definition of c^*, γ^* one finds that we have $2c_1(A) + |y| - |x| = 0$ ← "index zero problem"

$\Longrightarrow$ "dim" $= -2 < 0$ so $\emptyset$ set. ✓ ∎

↖ intuition: cannot be an $\underline{\text{isolated}}$ solution because we can rotate sphere:

(moral reason!)