Symplectic cohomology via circle-actions, and generation results for Fukaya categories.

The cohomological McKay correspondence via Floer theory.

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The big picture

Symplectic manifolds are locally $(\mathbb{C}^n, \omega_0)$, so we seek global invariants.

	M **closed**	M **open** or **closed with** ∂M
"closed strings"	$HF^*(H) \cong QH^*(M)$ Floer/Quantum cohomology	$SH^*(M)$ Symplectic cohomology
"open strings"	$HF^*(L_1, L_2)$ Lagrangian Floer cohomology	$HW^*(L_1, L_2)$ Wrapped Floer cohomology

Lots of algebraic structure: $HF^*(L_1, L_2)$ *are* $QH^*(M)$*-modules.*

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Lots of algebraic structure: $HF^*(L_1, L_2)$ *are* $QH^*(M)$*-modules.*

Fukaya category $\mathcal{F}(M)$: package all Lagrangians $L \subset M$ up into A_∞-category, $\mathrm{Mor}(L_1, L_2) =$ chain complex underlying $HF^*(L_1, L_2)$.

Wrapped Fukaya category $\mathcal{W}(M)$: allow non-compact L, use HW^*.

Homological Mirror symmetry (Kontsevich '94): Often have mirror pairs (X, J) complex variety and (M, ω) symplectic manifold:

Category of Coherent Sheaves on X	$\xleftrightarrow[\text{derived categories}]{\text{Equivalence on the}}$	Fukaya category $\mathcal{F}(M)$ of M

Loosely, relate Lagrangians $L \subset M$ to holo vector bundles $V \to X$.

Closed-open string map: $QH^*(M) \to \mathrm{HH}^*(\mathcal{F}(M)) \cong \mathrm{HH}^*(D^b\mathrm{Coh}(X))$, and $SH^*(M) \to \mathrm{HH}^*(\mathcal{W}(M)) \cong \mathrm{HH}^*(D^b\mathrm{Coh}(X))$. Sometimes isos.

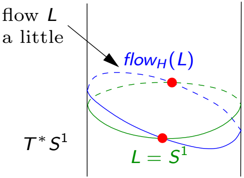
The big picture in a little picture

Example 1: $X = \mathbb{C}^*$ and $M = T^*S^1$

$D^b\mathrm{Coh}(X)$ generated by structure sheaf $\mathcal{O}$, $\mathrm{Mor}(\mathcal{O}, \mathcal{O}) = \mathbb{K}[X] = \mathbb{K}[x, x^{-1}]$.

$D^\pi\mathcal{F}(M)$ gen. by L = 0-section

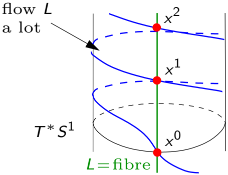
$\mathrm{Mor}(L, L) = \mathbb{K} \oplus \mathbb{K} \simeq C_{1-*}(S^1)$



$QH^*(M) \cong H^*(M) \cong \mathbb{K} \oplus \mathbb{K}$

$D^\pi\mathcal{W}(M)$ gen. by L = fiber $\subset T^*S^1$

$\mathrm{Mor}(L, L) = \mathbb{K}[x, x^{-1}] \simeq C_{1-*}(\Omega S^1)$



$SH^*(M) \cong H_{1-*}(\mathcal{L}S^1) \cong \mathbb{K}[x, x^{-1}] \otimes H^*(S^1)$

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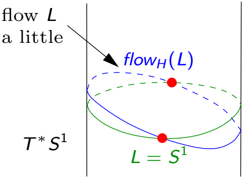

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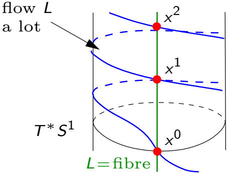
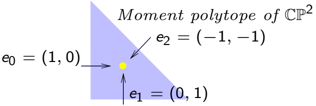


$SH^*(M) \cong H_{1-*}(\mathcal{L}S^1) \cong \mathbb{K}[x, x^{-1}] \otimes H^*(S^1)$

Example 2: $X = \mathbb{CP}^2$ and $M =$ Landau-Ginzburg model $((\mathbb{C}^*)^2, W)$

$D^b\mathrm{Coh}(X)$ generated by 3 vector bundles $\mathcal{O}, \mathcal{O}(-1), \mathcal{O}(-2)$



$W = z_1 + z_2 + z_1^{-1}z_2^{-1} : (\mathbb{C}^*)^2 \to \mathbb{C}$

$\mathcal{F}(M, W) =$ "Fukaya-Seidel category"

$D^b(\mathcal{F}(M, W))$ generated by 3 objects

$L = S^1 \times S^1$ with 3 "holonomies" $\in H^1(L; \mathbb{C})$

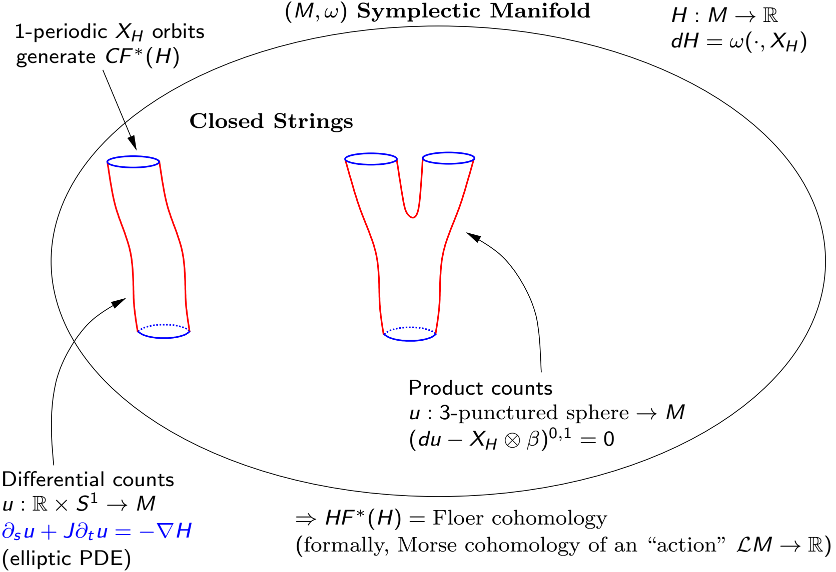
$\leftrightarrow$ 3 Critical points of W.

$M = \mathbb{CP}^2$: "$D^\pi(\mathcal{F}(M)) \cong H^0(\mathcal{MF}(W))$", $\mathcal{MF}(W) =$ Cat. Matrix Factorizations.

Actually pieces $\mathcal{F}_\lambda(M), \mathcal{MF}(W - \lambda)$: $QH^*(M) = \mathbb{K}[x]/(x^3 - \lambda^3) \cong \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$.

For non-compact M, expect $D^\pi(\mathcal{F}(M)) \cong D^b\mathrm{Perf}(X)$ for singular variety X.

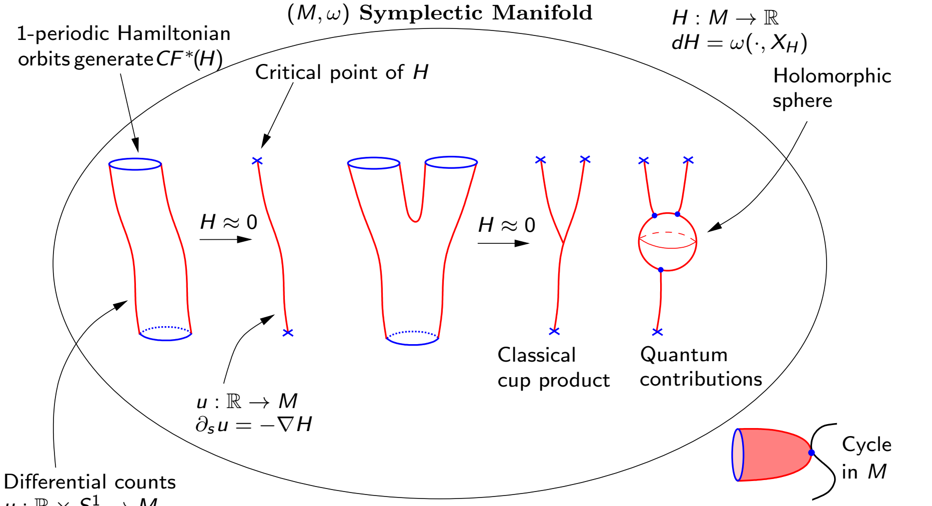
Floer, Quantum and Symplectic Cohomology



$\Rightarrow HF^*(H) =$ Floer cohomology
(formally, Morse cohomology of an "action" $\mathcal{L}M \to \mathbb{R}$)

M compact $\Rightarrow QH^*(M) \cong HF^*(H)$ for any H (Floer 1989, PSS 1996)

Floer, Quantum and Symplectic Cohomology



$\Rightarrow HF^*(H) \cong$ Morse cohomology $\cong H^*(M)$ as a vector space

$\Rightarrow HF^*(H) \cong$ Quantum cohomology $QH^*(M)$ as a ring

M non-compact: $QH^*(M) \cong HF^*(H_{\text{small}}) \to SH^*(M) = \varinjlim HF^*(H)$ ring hom (R.'12)

$(M \setminus \{\text{compact}\}, \omega) \cong (\Sigma \times (1, \infty), d(R\alpha))$ for contact mfd (Σ, α); H linear in R at ∞

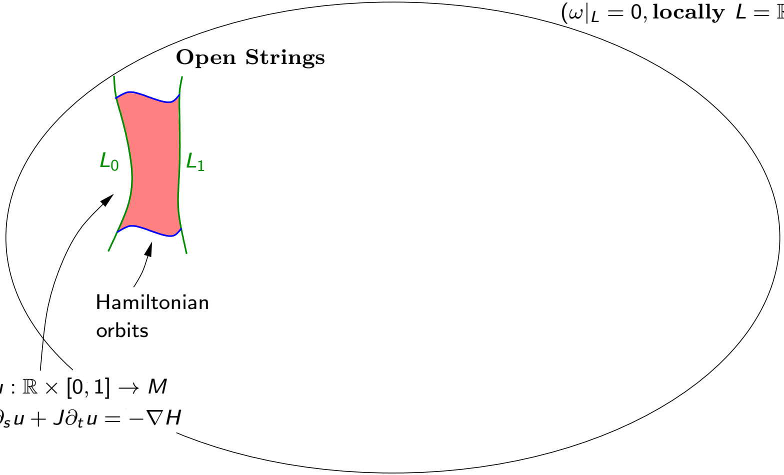
Examples

- $SH^*(\mathbb{C}^n) = 0$. Also (Cieliebak 2002): $SH^*(\text{subcrit. Stein mfd}) = 0$
- $\widetilde{\mathbb{C}^n} = \mathbb{C}^n$ blown up at 0, (R. 2013):
  $SH^*(\widetilde{\mathbb{C}^n}) = QH^*(\widetilde{\mathbb{C}^n})/(\text{generalised 0-espace of } \omega) \cong \mathbb{K}[\omega]/(\omega^n + t)$
  $\mathbb{K} =$ formal Laurent series in t over a field.
- In the Fano regime $1 \leq k \leq m$, (R. 2013):
  $$\begin{aligned} SH^*(\mathcal{O}_{\mathbb{P}^m}(-k)) &\cong \mathbb{K}[\omega]/(\omega^{1+m-k} - t(-k)^k) \\ QH^*(\mathcal{O}_{\mathbb{P}^m}(-k)) &\cong \mathbb{K}[\omega]/(\omega^{1+m} - t(-k)^k\omega^k) \\ \text{compare: } QH^*(\mathbb{P}^m) &\cong \mathbb{K}[\omega]/(\omega^{1+m} - t) \end{aligned}$$
- $SH^*(T^*N) \cong H_{n-*}(\mathcal{L}N)$ (Viterbo 1996)
  (also: Abbondandolo-Schwarz 2004, Salamon-Weber 2003)
- $\pi : E \to B$ negative vector bundle over sympl.mfd., (R. 2013):
  $SH^*(E) \cong QH^*(E)_{[B]} \cong$
  $QH^*(E)/(\text{generalised 0-eigensummand of } \pi^* c_{\text{top}}(E))$
- M compact toric Fano: $\mathrm{Jac}(W) \cong QH^*(M)$ (Batyrev 93/Givental 96)
  (R. 2015): For "many" non-compact Fano toric varieties:
  $\mathrm{Jac}(W) \cong SH^*(M) \cong QH^*(M)_{\mathrm{PD}[D_1], \ldots, \mathrm{PD}[D_r]}$ not $QH^*(M)$!

Fukaya and Wrapped Fukaya categories $\mathcal{F}(M), \mathcal{W}(M)$

(M, ω) **Symplectic Manifold**

$L_j \subset M$ **Lagrangian submanifolds**
$(\omega|_L = 0,$ **locally** $L = \mathbb{R}^n \subset \mathbb{C}^n = M)$



$u : \mathbb{R} \times [0, 1] \to M$

$\partial_s u + J \partial_t u = -\nabla H$

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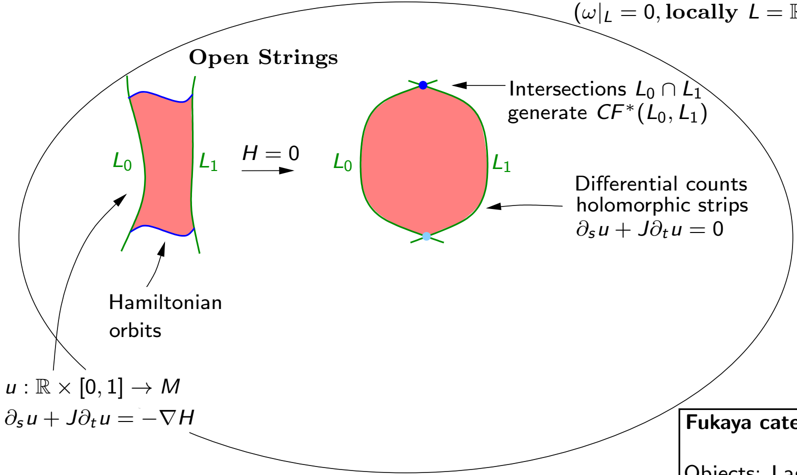

Fukaya category $F(M)$
Objects: Lagrangians
Morphisms: $CF^*(L_0, L_1)$

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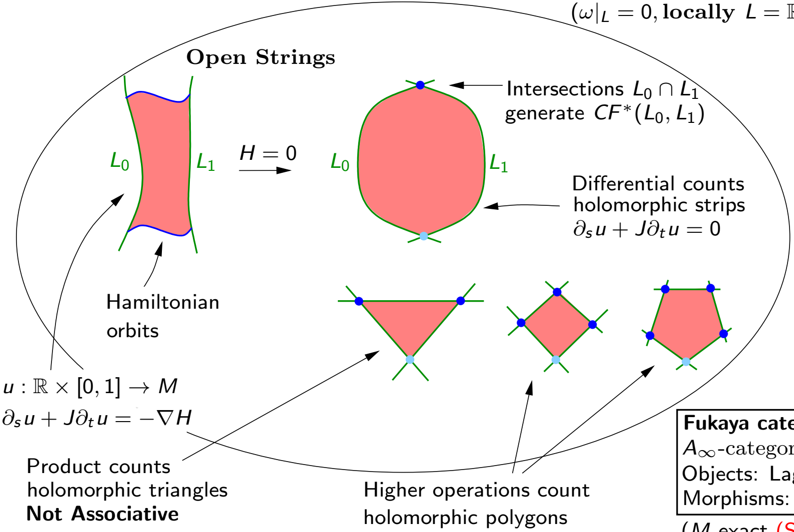


Fukaya category $F(M)$
A_∞-category
Objects: Lagrangians
Morphisms: $CF^*(L_0, L_1)$

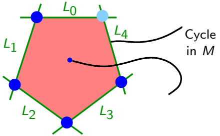
(M exact (Seidel 2008))

M non-cpt $\Rightarrow$ Wrapped cat. $\mathcal{W}(M)$ allow non-cpt Lags. Morphs: "$\varinjlim$" $CF^*(\varphi^1_H(L_0), L_1)$

(M exact (Fukaya-Seidel-Smith 2007 / Abouzaid 2010), M Fano (R./Smith 2012))

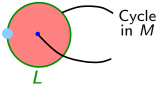
The open-closed and closed-open string maps $\mathcal{OC}$, $\mathcal{CO}$

$\boxed{\mathcal{OC} : \mathrm{HH}_*(\mathcal{F}(M)) \to QH^*(M)}$ (String maps appeared in Seidel's ICM talk '02)

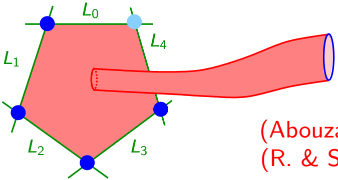


Here $\mathcal{OC}_4$ on Hochschild Homology bar complex,
$\underline{CF^*(L_4, L_0)} \otimes CF^*(L_3, L_4) \otimes \ldots \otimes CF^*(L_0, L_1) \to QH^*(M)$

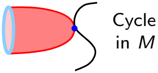
(0-part $\mathcal{OC}_0 : HF^*(L, L) \to QH^*(M)$ is:
$\mathcal{OC}_0$ by Albers '05, Biran-Cornea '08)



$\boxed{\mathcal{OC} : \mathrm{HH}_*(\mathcal{W}(M)) \to SH^*(M)}$ In particular, $\mathcal{OC}_0 : HW^*(L, L) \to SH^*(M)$



Get above picture if glue $HF^*(H) \cong QH^*(M)$



(Abouzaid 2010 in exact case)
(R. & Smith 2012-17 in monotone case)

"Dually" $\mathcal{CO} : QH^*(M) \to HH^*(\mathcal{F}(M))$ and $SH^*(M) \to HH^*(\mathcal{W}(M))$.

e.g. counts of the picture above defines the following factor of HH^4:

$\mathrm{Hom}\left(CF^*(L_3, L_4) \otimes CF^*(L_2, L_3) \otimes CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \,,\, \underline{CF^*(L_0, L_4)} \right)$

Generation Criterion (Abouzaid exact '10, R./Smith monotone '17)

Restrict $\mathcal{OC}$ to a subcategory generated by $L_1, \ldots, L_n$, then:

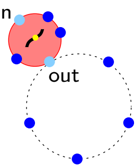
If $\mathcal{OC}$ hits $1 \Rightarrow L_1, \ldots, L_n$ split-generate whole category.

Module structure

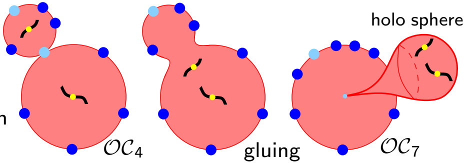
Theorem (R. & Smith '12-'17, independently Ganatra '13 for exact M)

- $\mathrm{HH}_*(\mathcal{F}(M))$ *is* $QH^*(M)$*-module*
- $\mathrm{HH}_*(\mathcal{W}(M))$ *is* $SH^*(M)$*-module*
- $\mathcal{OC}$ *is a* $QH^*(M)$*-module hom, respectively an* $SH^*(M)$*-module hom*
- $\mathcal{CO}$ *is a unital algebra homomorphism.*

Here $QH^*(M)$ acts on $\mathrm{CC}_7(\mathcal{F}(M))$ lands in CC_4



Why $\mathcal{OC}$ is $QH^*(M)$-mod homomorphism



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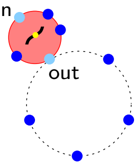

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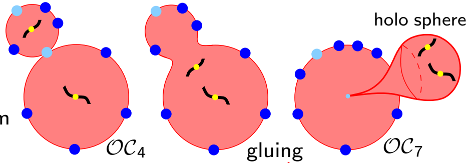

Monotonicity and c_1-eigenvalues (Kontsevich, Seidel, Auroux)

Monotone Lagrangians $L \subset$ monotone M with $HF^*(L,L) \neq 0$, the unit $[L] \in HF^*$ satisfies $c_1(TM) * [L] = \lambda [L]$, $\quad \lambda \in \{\text{evalues of } c_1(TM) \in QH^*\}$

In fact, to ensure (Floer differential)$^2 = 0$, restrict to $\mathcal{F}_\lambda(M) = \{\text{only such } L\}$.

Eigensummand decomposition (R./Smith) $\oplus \mathcal{OC}_\lambda : \mathrm{HH}_*(\mathcal{F}_\lambda(M)) \to QH^*(M)_\lambda$

Corollary Hitting invertible in $QH^*(M)_\lambda \Rightarrow$ Generation for $\mathcal{F}_\lambda(M)$.

Example. If eigensummands $QH^*(M)_\lambda$ are 1-dimensional (so field!) then:

$$\mathcal{OC}_\lambda \text{ non-zero} \Rightarrow \text{hit invertible} \Rightarrow \text{Generation for } \mathcal{F}_\lambda(M)$$

Applications to Fano toric varieties

$QH^*(\mathbb{CP}^2) = \mathbb{K}[x]/(x^3 - t) = \frac{\mathbb{K}[x]}{x-1t} \oplus \frac{\mathbb{K}[x]}{x-\zeta t} \oplus \frac{\mathbb{K}[x]}{x-\zeta^2 t}$ $\qquad \zeta = e^{2\pi i/3}$

Trick: $[pt] \in C_*(L) \simeq CF^*(L,L)$, leading $\mathcal{OC}([pt])$ term is constant disc,

$\boxed{\mathcal{OC}([pt]) = \mathrm{PD}(\mathrm{point}) + \text{higher } t}$ $\Rightarrow$non-zero$\Rightarrow$generation if $\exists L, \mathfrak{d}[\mathrm{pt}] = 0$

Key: (Cho-Oh'06) Crit$(W) \leftrightarrow$ tori L with $\lambda = W(z)$, and $\mathfrak{d}[\mathrm{pt}] = 0$.

$W = Z_1 + Z_2 + tZ_1^{-1}Z_2^{-1}$ has 3 crit points, crit vals = three evals of c_1.

Batyrev'93/Givental'96: $QH^*(M) \cong \mathrm{Jac}(W) = \mathbb{K}[z_1^{\pm 1}, \ldots]/(\partial_{z_1} W, \ldots)$, $c_1(M) \mapsto W$

$\Rightarrow$ trick works for closed Fano M, Morse W_M. But don't need Morse by Cho-Hong-Lau'19 & Lekili-Evans'19. Don't need Fano by Abouzaid-FOOO

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Theorem (R./Smith '12-'17, R.'16)

$\mathcal{W}(\mathcal{O}_{\mathbb{P}^m}(-k))$ for $1 \leq k \leq m$ is split-generated by Lagrangian torus $\mathcal{L}$ with $1 + m - k$ choices of holonomy. ($\mathcal{L}$ = lift Clifford torus to sphere bundle)

Sketch Proof. $SH^*(M) = \Lambda[\omega]/(\omega^{1+m-k} - (-k)^k t) \cong \mathrm{Jac}(W)$, and

$\boxed{\mathcal{OC}([pt]) = (-k\mu)t \cdot \mathrm{PD}(\mathrm{fiber}) + \mathcal{O}(t^2) \neq 0}$

(leading term: disc in fibre $\cong \mathbb{C}$ bounding S^1, it hits $[\mathbb{P}^m]$ in 1 point) □

Theorem. (R.'16) Works for any monotone toric negative line bundle $E \to B$ with W_B Morse. **Key ingredient** R.'16: $SH^*(E) = \mathrm{Jac}(W_E)$.

A message from our sponsor: Technicalities

Fukaya-Oh-Ohta-Ono over the years have carried out major foundational work on Floer theory: no assumptions on M (closed sympl.), use Kuranishi structures. Instead we use non-compact M, use explicit perturbations of auxiliary data, but require assumptions on L, M. At ∞: $\omega = d(R\alpha)$, L "conical" (Legendrian$\times\mathbb{R}$).

"Exact" means: $\omega = d\theta$ globally on M, exact Lags L.

1. Well-defined (single-valued) action functionals for Floer theory!
2. Easy energy estimates, no holo curves, no bubbling problems
3. e.g. T^*N and (Wein)Stein manifolds, but no interesting Kähler mfds
4. Can avoid direct limits: use Hamiltonians quadratic in R in A_∞-category:

$$CF(\varphi_H^1(L), L) \otimes CF(\varphi_H^1(L), L) \equiv CF(\varphi_H^1(L), L) \otimes CF(\varphi_H^2(L), \varphi_H^1(L)) \xrightarrow{\mu^2} CF(\varphi_H^2(L), L)$$

Abouzaid '10: canonical $CF(\varphi_H^2(L), L) \cong CF(\varphi_H^1(L), L)$ via ∂_R-flow (Liouville)

"Monotone": $c_1(M) = k\,\omega$, $k > 0$, orientable monotone L ($\omega(u) = \mathrm{Maslov}(u)/2\lambda$ for discs)

1. Bubbling controllable: $\omega(u) > 0 \Rightarrow c_1(u) > 0 \Rightarrow$ positive Fredholm index
2. Energy: Novikov ring formal variable t, high energy $\Rightarrow$ high t-power
3. Interesting mfds: negative line bundles over closed Kähler mfds, blow-ups
4. Must use direct limit over Hamiltonians linear in R in A_∞-category

A message from our sponsor: Technicalities (brace yourself)

Key issue: implement the direct limit at the chain level.
Exact: A_∞-algebra $CW^*(L,L)$ of one Lagrangian: Abouzaid-Seidel '10.
Monotone: A_∞-category: R.-Smith '17 (works also for Exact).

Fix $H : M \to \mathbb{R}$, linear at ∞.

$$CW^*(L_i, L_j) = \bigoplus_{w=1}^{\infty} CF^*(L_i, L_j; wH)[\mathbf{q}]$$

- $CF^*(L_i, L_j; wH)$ generated by 1-orbits of X_{wH} from L_i to L_j, the "chords".
- $\mathbf{q}$ formal variable of degree -1 satisfying $\mathbf{q}^2 = 0$.
  $\Rightarrow$ *Two* copies $CF^*(L_i,L_j;wH)[\mathbf{q}] = CF^*(L_i,L_j;wH) \oplus CF^*(L_i,L_j;wH)\mathbf{q}$.

Differential: $$\mu^1(x + \mathbf{q}y) = (-1)^{|x|}\mathfrak{d}x + (-1)^{|y|}(\mathbf{q}\mathfrak{d}y + \mathfrak{K}y - y)$$

- $\mathfrak{d} : CF^*(L_i, L_j; wH) \to CF^{*+1}(L_i, L_j; wH)$ usual Floer differential

$\mathfrak{d}$ counts strips u bounding L_i, L_j, asymptotic to chords, $du - wX_H \otimes dt$ holo.

- $\mathfrak{K} : CF^*(L_i, L_j; wH) \to CF^*(L_i, L_j; (w+1)H)$ Floer continuation map.
- if $\mathfrak{d}(y) = 0$ then $\mathbf{q}y$ identifies y and $\mathfrak{K}y$ at the cohomology level.

Cohomology direct limit: $[y] = [\mathfrak{K}y] \in \varinjlim HF^*(\varphi^1_{wH}(L), L) = HW^*(L,L)$.

- subcomplex $(\partial_{\mathbf{q}} = 0)$ yields representatives of $\oplus HF^*(L_i, L_j; wH)$.

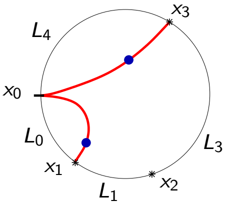
Analogously for symplectic cohomology: $SC^*(M) = \oplus CF^*(wH)[\mathbf{q}]$

A message from our sponsor: Technicalities (there's more?)

Rough idea of how one counts the Floer PDE solutions:

$(du - X_H \otimes \gamma)^{0,1} = 0$ for u : (decorated disc with bdry punctures) $\to M$.

- γ = 1-form on punctured disc
- $\gamma = w_i\, dt$ near input puncture for $x_i \in CF^*(L_{i-1}, L_i; w_i H)$ (local strip-like coords)
- **Crucial**: $d\gamma(\cdot, J\cdot) \leq 0$ so a max principle stops solutions going to ∞
- Stokes's theorem $\Rightarrow 0 \leq -\int d\gamma = w_0 - \sum_{\text{inputs}} w_i$ (so need a big output weight w_0!)



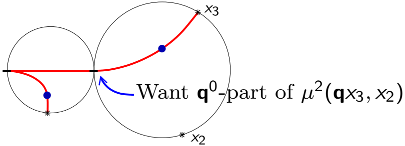
Picture: $\pm t^{\text{Energy}} x_0$ contribution to A_∞-map

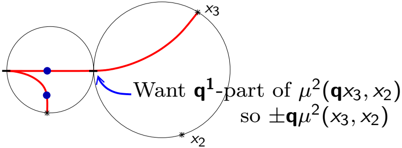
$\mu^3(\mathbf{q}x_3 \otimes x_2 \otimes \mathbf{q}x_1) \in CF^*(L_0, L_4; w_0 H)$

- x_j has $\mathbf{q}$ $\leftrightarrow$ (geodesic $x_0 x_j$ has marker) $\leftrightarrow \exists \beta_j$
- $\gamma = w_1\alpha_1 + w_2\alpha_2 + w_3\alpha_3 + \beta_1 + \beta_3$
- $\alpha_i = dt$ near x_0, x_i, else 0 at bdry; $d\alpha_i = 0$
- $\beta_j = dt$ near x_0, else 0 at bdry; $d\beta_j \leq 0 \neq 0$ only near marker
- $w_0 = w_1 + w_2 + w_3 + 1 + 1$, due to β_1, β_3

- $\mathbf{q}x_0$-output: determined by asking μ^3 is $\partial_{\mathbf{q}}$-linear. Geometrically it corresponds to one of the markers escaping to x_0.

Example A_∞-eqns:





The role of monotonicity

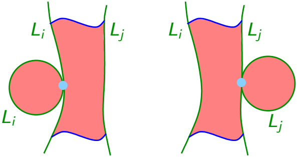
• For (Fredholm) index 0 solution counts, bubbling is not an issue: non-constant bubbles have positive area so positive index.

$\Rightarrow$ main component of the broken solution would have $\mathrm{virdim} < 0$.

• Proof $\mathfrak{d}^2 = 0$: index 2 solutions $\Rightarrow \exists$ Maslov 2 (Chern 1) J-holo sphere bubble? No: generic $J \Rightarrow \{$ spheres $\}$ smooth moduli space $\mathrm{codim}_{\mathbb{R}} = 4$

• Proof $\mathfrak{d}^2 = 0$: $\exists$ Maslov 2 disc bubble with boundary on one L?
Lazzarini '10 $\Rightarrow \{$ discs $\}$ smooth moduli space. (2 is min Maslov: L orientable)
Key: Moduli space of Maslov 2 discs with boundary marked point, evaluation at marker = locally finite (dim L)-cycle so a multiple of top class $[L]$,

$$\mathfrak{m}_0(L) = \sum t^{\omega[\beta]}\, \mathrm{ev}_*[\mathcal{M}_1(\beta)] = m_0(L)\,[L] \in C^{\mathrm{lf}}_{\dim(L)}(L; \mathrm{NovikovRing}).$$



Oh '93/'95 $\Rightarrow \mathfrak{d} \circ \mathfrak{d}(x) = (m_0(L_i) - m_0(L_j))\, x$.
Serious problem!
Conclusion: Break up A_∞-category so $\mu^1 \circ \mu^1 = 0$:

$\mathcal{F}_\lambda(M)$: only allow L with $m_0(L) = \lambda$

Compare: Cat of Matrix Factorizations, $\partial^2\big(f{:}\mathcal{MF}(W-\lambda)\to\mathcal{MF}(W-\lambda')\big)=(\lambda-\lambda')f$.
CONVENTION: from now on $\mathcal{F}(M), \mathcal{W}(M)$ means $\mathcal{F}_\lambda(M), \mathcal{W}_\lambda(M)$.

The λ are eigenvalues of $c_1(M)\cdot : QH^*(M) \to QH^*(M)$.

Kontsevich, Seidel and Auroux (Auroux '07):

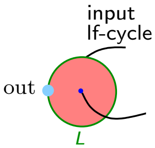
$$HF^*(L,L) \neq 0 \Rightarrow m_0(L) \text{ is an eigenvalue of } c_1(M).$$

1$\star$ Suppose L disjoint from lf-cycle D representing $c_1(M)$. (so $PD(c_1(M)) = D$)

2$\star$ Suppose MaslovIndex(J-holo $u : (\mathbb{D}, \partial\mathbb{D}) \to (M, L)$) $= 2\#(u \cap D)$.

$\Rightarrow$ discs counted by $m_0(L)$ hit D once, reparametrise: $u(0) \in D$

Use unital ring homomorphism $\mathcal{CO} : QH^*(M) \to HF^*(L,L)$:



$\mathcal{CO}(D) =$ "(the discs u above)" $= m_0(L)\,[L]$.

$\mathcal{CO}(\text{unit } [M]) =$ "(constant discs)" $= [L]$. (Maslov 0 discs)

$\Rightarrow \mathcal{CO}(c_1(M) - m_0(L)[M]) = 0$ not invertible! (unit $[L] \neq 0$)

Finally: unital ring hom sends invertibles $\mapsto$ invertibles.

Claim (R.-Smith '17): $\star$ conditions hold for us.

Proof: MaslovIndex = homological intersection number with PD of Maslov cycle $\mu_L \in H^2(M,L)$ (dualise MaslovIndex: $H_2(M,L) \to \mathbb{Z}$).

Also $\mu_L \mapsto 2c_1(M)$ via $H^2(M,L) \to H^2(M)$. Recall $PD(c_1(M)) =$ zero locus of generic smooth section s of a complex line bundle $\mathcal{E}$ on M with $c_1(\mathcal{E}) = c_1(M)$.

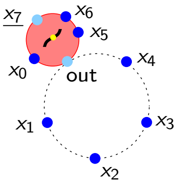
But $c_1(M)|_L = k\omega|_L = 0 \Rightarrow \mathcal{E}$ trivial near $L \Rightarrow$ can ensure $s \neq 0$ near L $\square$

Eigensummand decomposition of the string maps

Let $c = c_1(M) - \lambda\,\mathrm{id}$

Let $QH^*(M)_\lambda = \ker c^{\mathrm{large}}$ = generalised λ-eigensummand of $c_1(M)$

Sketch proof that $\mathcal{OC} : \mathrm{HH}_*(\mathcal{F}_\lambda(M)) \to QH^*(M)_\lambda$:



Picture: QC^*-action ψ_c on $\underline{x_7} \otimes x_6 \otimes \cdots \otimes x_0 \in \mathrm{CC}_7$, showing contribution $\underline{\psi_c(x_0 \otimes \underline{x_7} \otimes x_6 \otimes x_5)} \otimes x_4 \otimes \cdots \otimes x_1 \in \mathrm{CC}_4$, in picture: $\underline{\mathrm{out}} \otimes x_4 \otimes x_3 \otimes x_2 \otimes x_1 \in \mathrm{CC}_4$.

"Length of word" keeps decreasing if keep applying ψ_c, unless apply ψ_c to just one element so hit $CF^*(L, L)$. But ψ_c on $HF^*(L, L)$ is μ^2-product by $\mathcal{CO}(c) = 0$ (previous slide)

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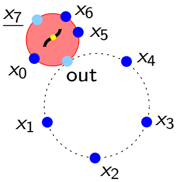

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Acceleration Diagram (R/Smith'17)

$$\begin{array}{ccc} \mathrm{HH}_*(\mathcal{F}_\lambda(M)) & \xrightarrow{\mathrm{HH}_*(\mathcal{AF})} & \mathrm{HH}_*(\mathcal{W}_\lambda(M)) \\ \downarrow \mathrm{OC} & & \downarrow \mathrm{OC} \\ QH^*(M)_\lambda & \xrightarrow{c^*} & SH^*(M)_\lambda \end{array}$$

Not as simple as it looks! Cannot allow $w = 0$ in $CW^* = \oplus CF^*(L_0, L_1; wH)$.

New A_∞-category $\mathcal{W}_\diamond(M)$: for *compact* Lags L_0, L_1, extra summand $CF^*(L_0, L_1)[\mathbf{q}]$

(perturb $CF^*(\varphi_K^1(L_0), L_1)$ by compactly supported K as in Seidel).

Also $SC^*_\diamond = QC^*(M)[\mathbf{q}] \oplus SC^*$.

The natural functor $\mathcal{W}(M) \to \mathcal{W}_\diamond(M)$ is a quasi-isomorphism. In an A_∞-category one can always invert quasi-isos. So:

$$\mathcal{AF} : \mathcal{F}(M) \xrightarrow{include} \mathcal{W}_\diamond(M) \xrightarrow{quasi-iso} \mathcal{W}(M).$$

Rebooting...

Generators and relations for $SH^*(M)$ for toric M

Seidel representation

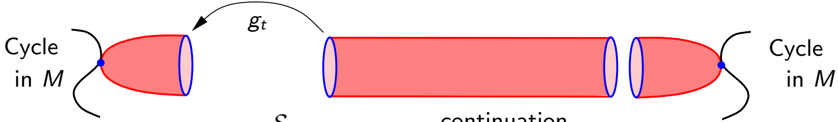
Theorem (Seidel 1997)

There is a representation $\mathcal{S} : \pi_1 \widetilde{\mathrm{Ham}}(M) \to \mathrm{Aut}(QH^*(M))$ *where* $\mathcal{S}(\widetilde{g}) =$ *quantum product by an invertible element* $\mathcal{S}(\widetilde{g})(1)$.

At the chain level: $\mathcal{S} : CF^*(H) \xrightarrow{\text{identification}} CF^*(g^*H)$

$$\widetilde{x} \longmapsto \widetilde{g}^{-1} \cdot \widetilde{x}$$

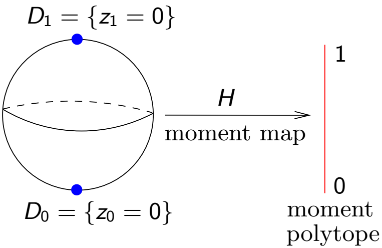
$g^*H = H \circ g - K_g \circ g$ ensures $g^* d\mathbb{A}_H = d\mathbb{A}_{g^*H}$ ($\mathbb{A} =$ Floer action), thus generators and moduli spaces are identified. As $HF^*(H) \cong QH^*(M)$ independently of H, one gets an automorphism of $QH^*(M)$:



$$QH^*(M) \longrightarrow HF^*(H_0) \xrightarrow[\cong]{\mathcal{S}} HF^*(H_{-1}) \xrightarrow{\text{continuation}} HF^*(H_0) \longrightarrow QH^*(M)$$

Remark. $\mathcal{S}(\widetilde{g})$ can be phrased as a 2-point GW-invariant counting holomorphic sections of a bundle over S^2, fibre M, transition g.

Example: $M = \mathbb{CP}^1$



Hamiltonian S^1-actions which rotate about the toric divisors $D_j = \{z_j = 0\}$

$$g_0(t)z = [e^{2\pi i t} z_0 : z_1]$$
$$g_1(t)z = [z_0 : e^{2\pi i t} z_1]$$

determine invertibles in $QH^*(\mathbb{CP}^1)$:

$$x_0 = \mathcal{S}(\widetilde{g}_0)(1) = \mathrm{PD}[D_0] = \omega$$
$$x_1 = \mathcal{S}(\widetilde{g}_1)(1) = \mathrm{PD}[D_1] = \omega$$

$$[\lambda z_0 : z_1] = [z_0 : \lambda^{-1} z_1] \Rightarrow \widetilde{g}_0 = \widetilde{g}_1^{-1} \cdot t$$
$$\Rightarrow x_0 = \mathcal{S}(\widetilde{g}_0) = \mathcal{S}(\widetilde{g}_1^{-1} \cdot t) = x_1^{-1} \cdot t$$
$\Rightarrow x_0 x_1 = t$, therefore $\omega * \omega = t$.

Theorem (McDuff-Tolman 2006)

For closed Fano toric symplectic manifolds, the relations among the S^1-rotations around the toric divisors $D_j = \{z_j = 0\}$ yield, via $\mathcal{S}$, the non-classical relations among the $x_j = \mathrm{PD}[D_j]$.

$$QH^*(M) = \mathbb{K}[x_0, x_1, \ldots] / \begin{pmatrix} \textit{homology relations among } x_j = \mathrm{PD}[D_j] \\ \mathcal{S}(\textit{relations among rotations about } D_j) \end{pmatrix}$$

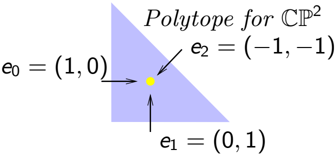
BATYREV (1993) / GIVENTAL (1996) / CIELIEBAK-SALAMON (2002) / McDUFF-TOLMAN (2004).

Landau-Ginzburg Superpotential W

Moment polytope $\Delta = \{y \in \mathbb{R}^m : \langle y, e_i \rangle \geq \lambda_i\}$

$W : (\mathbb{C}^*)^n \to \mathbb{K} = \text{Novikov Ring}$

$W(Z_1, \ldots, Z_m) = \sum t^{-\lambda_j} Z^{e_j}$



Example. $\mathbb{CP}^2$, $W = Z_1 + Z_2 + tZ_1^{-1}Z_2^{-1}$.

By Batyrev (1993):

$$QH^*(M) = \frac{\mathbb{K}[x_0, x_1, \ldots, x_r]}{\left(\substack{\text{linear relations} \\ \text{SR-relations}}\right)} \quad \cong \quad \mathrm{Jac}(W) = \frac{\mathbb{K}[Z_1^{\pm 1}, \ldots, Z_m^{\pm 1}]}{(\partial_{Z_1} W, \ldots, \partial_{Z_m} W)}$$

$$x_j = \mathrm{PD}[D_j] \quad \mapsto \quad t^{-\lambda_j} Z^{e_j}$$

$$\text{linear relations} \quad \to \quad \text{relations } \partial_{Z_i} W = 0.$$

The kernel includes the SR-relations since these correspond to (primitive) relations among edges, so relations among the Z^{e_i}.

Remark. The Z_j are automatically invertible in $\mathrm{Jac}(W)$.
The $x_j = \mathcal{S}(\widetilde{g}_j)$ are invertible because $g_j^{-1} \in \pi_1 \mathrm{Ham}(M)$.

Ostrover-Tyomkin'08 $p \in \mathrm{Crit}(W)$ non-degenerate $\Rightarrow$ field summand $\subset \mathrm{Jac}(W)$.

R.'16 Perturb $\omega \Rightarrow W$ Morse $\Rightarrow \mathrm{Jac}(W)$ becomes semi-simple $= \oplus$fields, so get Generation results for Fukaya/Wrapped Cat.

Generators and relations in $SH^*(M)$ from S^1-actions

For M non-compact, I constructed homs similar to the Seidel rep.

$$r : \pi_1 \widetilde{\mathrm{Ham}}_{\text{linear,slope}>0}(M) \to \mathrm{End}(QH^*(M))$$
$$\mathcal{R} : \pi_1 \widetilde{\mathrm{Ham}}_{\text{linear}}(M) \to \mathrm{Aut}(SH^*(M))$$

$(M \setminus \{\text{compact}\}, \omega) \cong (\Sigma \times (1, \infty), d(R\alpha))$ for contact mfd (Σ, α)

Theorem (R. '14)

If on $M \setminus \{compact\}$ the Reeb flow on Σ arises as a Hamiltonian S^1-action g on M, then there is an $r(g) \in QH^*(M)$ with

$$SH^*(M) = QH^*(M)_{r(g)} \qquad (localisation)$$

Theorem (R. '16)

For any non-compact Fano toric variety M (& technical conditions),

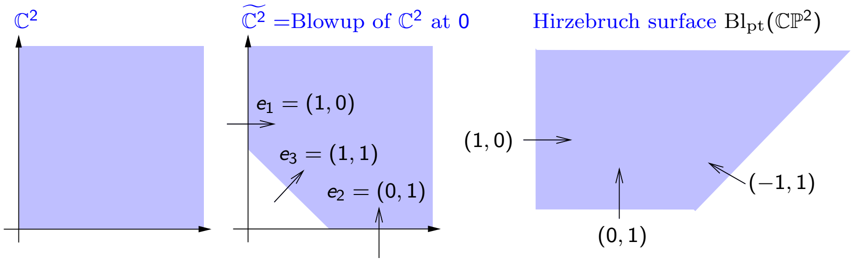
$$\begin{array}{ccccc} SH^*(M) & \cong & QH^*(M)_{\mathrm{PD}[D_1],\ldots,\mathrm{PD}[D_r]} & \cong & \mathrm{Jac}(W) \\ r(g_j) & \mapsto & \mathrm{PD}[D_j] & \mapsto & t^{-\lambda_j} z^{e_j} \end{array}$$

Example (R.'16) $E \to B$ Fano toric neg.line bdle.: $QH^*(E)$ vs $QH^*(B)$?

- same generators $x_0, \ldots, x_m$, same linear relations
- quantum relations: replace $t_B \mapsto t_E(-kx)^k \equiv t_E c_1(E)^k$

$$SH^*(E) \cong QH^*(E)_x \qquad (x = [\omega_E] = \pi^*[\omega_B],\ c_1[B] = \sum x_i)$$

Example: the blow-up of $\mathbb{C}^2$ at 0, namely $\mathcal{O}_{\mathbb{CP}^1}(-1)$



$W = z_1 + z_2 + t^{-1} z_1 z_2 \ \Rightarrow\ \partial_{z_1} W = 1 + t^{-1} z_2$, similarly for z_2

$\mathrm{Jac}(W) = \mathbb{K}[z_1^{\pm 1}, z_2^{\pm 1}]/(z_1 + t, z_2 + t) \cong \mathbb{K}$

$$QH^* = QH^*(\mathcal{O}_{\mathbb{P}^1}(-1)) = \mathbb{K}[x]/(x^2 + tx) \ncong \mathbb{K}$$

Agrees with Batyrev presentation:

SR-reln $e_1 + e_2 = e_3$, so $x_1 x_2 = x_3 \cdot t$

Linear reln $x_1 = x_2 = -x_3$. Put $x = x_1$.

SR-relation comes from the relation among rotations $\widetilde{g}_1 \widetilde{g}_2 = \widetilde{g}_3 \cdot t$.

Localize at x_j: $SH^* = \mathbb{K}[x^{\pm 1}]/(x^2 + tx) \cong \mathbb{K}[x]/(x + t) \cong \mathbb{K}$.

$SH^* \to \mathrm{Jac}(W)$, $x_1 \mapsto z_1 \equiv -t$, $x_2 \mapsto z_2 \equiv -t$, $x_3 \mapsto t^{-1} z_1 z_2 \equiv t$.

Symplectic vs Quantum when there is an S^1-action

$(M \setminus \{\text{compact}\}, \omega) \cong (\Sigma \times (1,\infty), d(R\alpha))$ for contact mfd (Σ, α)

Theorem (R. '14)

If on $M \setminus \{compact\}$ the Reeb flow on Σ arises as a Hamiltonian S^1-action g on M, then there is an $r(g) \in QH^*(M)$ with

$$SH^*(M) = QH^*(M)_{r(g)} \qquad (localisation)$$

Fix small H_0. Let $H_{k+1} = (g^{-1})^* H_k = H_k \circ g^{-1} + K \circ g^{-1}$ (K generates g)

(1) Canonically: $CF^*(H_k) \cong CF^*(H_{k+1})$, $x \mapsto g^{-1} \cdot x$ (Seidel 1997)

(2) $g \cdot [HF^*(H_k) \to HF^*(H_{k+1})] = [HF^*(H_{k+1}) \to HF^*(H_{k+2})]$

(3) $SH^*(M) = \varinjlim (QH \cong HF(H_0) \to HF(H_1) \to HF(H_2) \to \cdots)$

$\cong \varinjlim (QH \xrightarrow{*r} QH \xrightarrow{*r} QH \xrightarrow{*r} \cdots)$

$\cong QH^*(M)/(\text{generalized 0-espace of } r)$

(4) Description of $QH \xrightarrow{*r} QH$:

$QH^*(M) \longrightarrow HF^*(H_0) \xrightarrow[\text{canonical}]{\cong} HF^*(H_{-1}) \xrightarrow[\text{unknown continuation map}]{} HF^*(H_0) \longrightarrow QH^*(M)$

More precise statement of the toric presentation (R. '16)

Let X be a non-compact Fano toric manifold, such that the Hamiltonians generating the rotations g_j about the toric divisors satisfy the Floer theory maximum principle.

Let $\mathcal{J}$ = ideal generated by the linear and SR-relations. Then:

$QH^*(X) \cong \mathbb{K}[x_1, \ldots, x_r]/\mathcal{J}$, $\mathrm{PD}[D_j] \mapsto x_j$ (Batyrev presentation)
$SH^*(X) \cong \mathbb{K}[x_1^{\pm 1}, \ldots, x_r^{\pm}]/\mathcal{J}$, $r(g_j) \mapsto x_j$
$c^* : QH^*(X) \to SH^*(X)$ is the localization at $\mathrm{PD}[D_j]$.

$SH^*(X) \cong \mathrm{Jac}(W)$, $x_j \mapsto t^{-\lambda_j} z^{e_j}$.
$QH^*(X) \cong R_X/(\partial_{z_1} W, \ldots)$ for the $\mathbb{K}$-subalgebra $R_X \subset \mathbb{K}[x_1^{\pm 1}, \ldots]$ generated by z^e for $e \in \mathrm{Span}_{\mathbb{N}}(e_i)$.

$c_1(TX) = \sum \mathrm{PD}[D_j] = \sum x_j$ corresponds to $W \in \mathrm{Jac}(W)$.

Details about max principle: at infinity, want Hamiltonians to have the form $f(y) \cdot R$, where R is the radial coordinate, and $f : \Sigma \to \mathbb{R}$ is invariant under the Reeb flow. This is a slightly broader class of Hamiltonians than $k \cdot R$, and these can be used to define SH^*.

Rebooting...

The cohomological McKay Correspondence

Joint work with Mark McLean
Stony Brook University N.Y.

The big picture: resolutions of quotient singularities

Let $G \subset SL(n,\mathbb{C})$ be a finite subgroup $\neq 1$. Quotient $\mathbb{C}^n$ by G-action,

$$\boxed{X = \mathbb{C}^n/G}$$

$\Rightarrow \mathrm{Sing}(X) = \{[z] \in X : g \cdot z = z \text{ some } g \neq 1 \in G\}$.
$\Rightarrow X$ singular at 0 and possibly elsewhere. Take a resolution

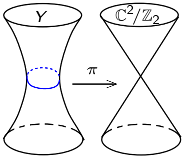
$$\boxed{\pi : Y \to X}$$

meaning: Y non-singular quasi-proj. var., π proper birational morphism, isomorphism away from the exceptional locus $E = \pi^{-1}(\mathrm{Sing}(X))$.

Question:

$$\{\textbf{Geometry of } Y\} \overset{?}{\longleftrightarrow} \{\textbf{Representation theory of } G\}.$$

Example A_1.



For $G = \{\pm I\} \subset SL(2,\mathbb{C})$, first embed

$$\nu_2 : \mathbb{C}^2/\{\pm I\} \hookrightarrow \mathbb{C}^3, \qquad (x,y) \mapsto (x^2, xy, y^2).$$

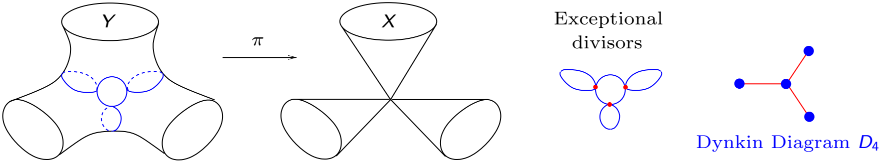
Image = Variety$(XZ - Y^2 = 0)$. Then blow-up 0 to get $Y = T^*\mathbb{CP}^1 = \mathcal{O}_{\mathbb{CP}^1}(-2)$. Generators of $H^*(Y) = \langle 1, \omega_{\mathbb{P}^1}\rangle$ $\leftrightarrow$ irreducible representations $1 \in GL(\mathbb{C})$ and $\pm 1 \in GL(\mathbb{C})$.

Classical McKay correspondence: dim $= 2$, $G \subset SL(2,\mathbb{C})$

Finite subgroups of $SL(2,\mathbb{C})$ are classified up to conjugation ($\mathbb{Z}_n$, $\tilde{\mathbb{D}}_{2n}$, $\tilde{\mathbb{T}}_{12}$, $\tilde{\mathbb{O}}_{24}$, $\tilde{\mathbb{I}}_{60}$), in 1 : 1 correspondence with *ADE* Dynkin Diagrams.

$\mathbb{C}[x,y]^G = \langle f_1, f_2, f_3 \rangle$ determine a surface $\mathbb{C}^2/G \hookrightarrow \mathbb{C}^3$ singular at 0.
Klein / 1934 Du Val: up to analytic isomorphism, such equations classify the **simple surface singularities** (rational double points).

Example: Quaternion group $\tilde{\mathbb{D}}_4 \subset SL(2,\mathbb{C}), X = \{x^2 + zy^2 + z^3 = 0\} \subset \mathbb{C}^3$:



In the **minimal** resolution $Y \to \mathbb{C}^2/G$, exceptional divisors E_i are in 1:1 correspondence with the non-trivial irreducible representations of G.
Remark: E_i generate $H_*(Y)$, $\#(\text{Irreducible Reps}) = \#(\text{Conj. Classes})$.

1980 McKay: McKay quiver for $\mathbb{C}^2$ is the extended Dynkin diagram.
1983 Gonzalez-Springberg, Verdier: K-theory $K_0(Y) \cong \text{Rep}(G)$.
2000 Kapranov, Vasserot: $D^b(\text{Coh}(Y)) \simeq D^b(\text{Coh}(\mathbb{C}^2)^G)$.

Higher dimensions: generalized McKay correspondence

Let $\pi : Y \to X = \mathbb{C}^n/G$ be a **crepant** resolution, so $K_Y = \pi^* K_X$ $(=0)$.

In general: $K_Y = \pi^* K_X + \sum a_i E_i$ for $a_i \geq 0$. Exceptional divisors E_i with $a_i = 0$ must appear on any resolution. Crepant resolutions may not exist.

Dixon-Harvey-Vafa-Witten '85 / Atiyah-Segal '89 / Hirzebruch-Höfer '90

Conjecture: $\chi(Y) = \#\mathrm{Conj.Classes}(G)$

Miles Reid '92 stated the **Cohomological McKay correspondence:**

$H^{\mathrm{odd}}(Y,\mathbb{C}) = 0$ and $\dim H^{2k}(Y,\mathbb{C}) = \#(\text{age k conjugacy classes})$

$g^{-1} \in \mathrm{Aut}(\mathbb{C}^n)$ has evalues $e^{ia_1}, \ldots, e^{ia_n}$, $a_j \in [0, 2\pi)$, define

$$\mathrm{age}(g) = \tfrac{1}{2\pi} \sum a_j \in [0, n).$$

Proofs: $\dim = 3$ Ito-Reid 1994, abelian G Batyrev-Dais 1996, in general Batyrev 1999 & Denef-Loeser 2002 (Motivic integration). Many ideas: Ito-Nakamura 1999, Ito-Nakajima 2000, Bridgeland-King-Reid 2001, ...

Open problem: find a "natural" basis for $H^*(Y,\mathbb{C}) \leftrightarrow \mathrm{Conj.classes}(G)$

Kaledin 2002: $\exists$ basis if $G \subset Sp(m) \subset SL(2m,\mathbb{C})$ (Valuations).

Nelson,et al. 2015: A_n-surface sing. $\Rightarrow \dim ESH^*_+(Y) = n+1 = |\mathrm{Conj}(\mathbb{Z}_{n+1})|$

Abreu-Macarini 2016: G abelian, $\mathbb{C}^n/G$ isolated $\Rightarrow \chi_{mean}(\mathrm{Link}) = \frac{1}{2}\chi(Y)$.

Main Theorem

Theorem (McLean - R. 2018)

Let $\mathbb{C}^n/G$ be an isolated singularity for $G \subset SL(n,\mathbb{C})$ a finite subgroup. Given any crepant resolution $\pi : Y \to \mathbb{C}^n/G$, there is a bijection

$$\mathrm{Conj}_k(G) = \{\textit{age } k \textit{ conjugacy classes}\} \to (\textit{basis of } H^{2k}(Y;\mathcal{K}))$$

and $H^{odd}(Y;\mathcal{K}) = 0$.

Rmk.1 Singularity at 0 is isolated if elements $\neq 1$ do not have eigenvalue 1. We are currently writing up the paper for the non-isolated case.

Rmk.2 Any field $\mathcal{K}$ of characteristic 0 works. For finite characteristic we need to assume char $\mathcal{K} \notin \{2, 3, \ldots, |G|\}$.

Key Idea: Build a $\mathbb{Z}$-graded symplectic invariant $SH^*_+(Y)$, and an iso

$$\boxed{\partial_{SC} \;:\; SH^{*-1}_+(Y) \cong H^*(Y)}$$

Generators are certain Hamiltonian orbits $x_g : S^1 \to Y$ inside Y, related to eigenvectors in $\mathbb{C}^n$ of the $g \in G$. Gradings:

$$\boxed{\mathrm{CZ}(x_g) - 1 = 2\,\mathrm{age}(g)}$$

Warm-up: Hamiltonian orbits in $X = \mathbb{C}^n/G$

Can assume $G \subset SU(n)$, by an averaging argument.
Diagonal $\mathbb{C}^*$-action on $\mathbb{C}^n$ descends to $X = \mathbb{C}^n/G$.
The S^1-action by e^{it} rotation is the Hamiltonian flow for $h = \frac{1}{2}\|z\|^2$.
Suppose $H_k : X \to \mathbb{R}$ convex function of h, so that flow on each slice

$$\mathcal{S} = \{\|z\| = \text{ constant } > 0\} \cong S^{2n-1}/G$$

is e^{iat} with a "speed" a that increases $\to k$ as we move to infinity in X.

What are the 1-periodic orbits?

Want $[e^{ia}z] = [z]$ in $\mathbb{C}^n/G$.
$\Leftrightarrow \quad e^{ia}z = g \cdot z$ for some $g \in G$.
$\Leftrightarrow \quad z$ is an e^{ia}-eigenvector of some $g \in G$.
Given an e^{ia}-eigenvector $z \in \mathbb{C}^n$ of $g \in G$ we get a 1-periodic orbit in $\mathcal{S}$:

$$\boxed{x_g(t) = e^{iat}z}$$

If G acts freely on $\mathbb{C}^n \setminus \{0\}$ (so $\mathbb{C}^n/G$ isolated) then from z we recover g uniquely, since z has no stabiliser. Thus orbits in $X \setminus \{0\}$ are uniquely labeled by elements of G. Only the conjugacy class $\{hgh^{-1} : h \in G\} \in \mathrm{Conj}(G)$ matters since identify $[z] = [h \cdot z]$ in X.

Hamiltonian orbits in Y

Key. Diagonal $\mathbb{C}^*$-action on $X = \mathbb{C}^n/G$ lifts to Y (uses Y crepant). Can pick Kähler form on Y so that the S^1-action is Hamiltonian, $h : Y \to \mathbb{R}$.

Floer theory. Pick $H_k : X \to \mathbb{R}$ increasing at infinity as $k \to \infty$.

Example 1: $H_k = (k+\varepsilon)\cdot h$

Example 2: $H_k = c_k(h)\cdot h$ for a cut-off c_k growing from 0 to $k+\varepsilon$.

$\partial_s u + J\partial_t u = -\nabla H$

Symplectic cohomology $SH^*(Y) = \varinjlim HF^*(H_k)$, where:
Floer complex: generators are the 1-periodic orbits of H_k.
Differential counts cylinders $u : \mathbb{R}\times S^1 \to Y$ connecting such orbits and satisfying a certain elliptic PDE (Floer's equation).

For $\mathbb{C}^n/G$ isolated, away from the exceptional divisor $E = \pi^{-1}(0)$, $Y \setminus E \cong X \setminus 0$ so we have the "same" 1-periodic orbits x_g arising in slices $\mathcal{S} \cong S^{2n-1}/G$ as for X. (When not isolated $\exists$ several lifts of x_g to Y, related to eigenvectors $z \in \mathrm{Sing}(X)$ having non-trivial stabilisers)

McLean-R. (mimicking R.2010): $SH^*(Y) = 0$ by a grading trick.

Compare: $SH^*(\mathbb{C}^n) = 0$ because the only 1-periodic orbit 0 for $H = (k+\varepsilon)\frac{1}{2}\|z\|^2$ has Conley-Zehnder index $\to -\infty$ as $k \to \infty$.

Positive Symplectic Cohomology SH^*_+

Aim. Only care about orbits in $\mathcal{S}$-slices, ignore constant orbits over 0. We want to kill the Morse subcomplex of orbits living over 0.

McLean-R. Build a new filtration allowing generalisation of Viterbo '96: Use $H_k = c_k(h) \cdot h$, have SES: $0 \to CF^*(H_0) \to CF^*(H_k) \to CF^*_+(H_k) \to 0$,

$$\cdots \to H^*(Y) \to SH^*(Y) \to SH^*_+(Y) \to H^{*+1}(Y) \to \cdots$$

For our resolution, $SH^*(Y) = 0$ so

$$\boxed{SH^{*-1}_+(Y) \cong H^*(Y)}$$

McLean-R. Our filtration also yields a Morse-Bott spectral sequence (like Morse-Bott spectral sequence in exact case in Kwon - van Koert '16) : $\mathcal{O}_{\mathbf{g},a} :=$ moduli space of 1-orbits associated to eigenvectors $[z] \in \mathbb{C}^n/G$ with eigenvalue e^{ia}, for each conjugacy class $\mathbf{g} \in \mathrm{Conj}(G)$. Then

$$\bigoplus_{\mathbf{g}\in\mathrm{Conj}(G),\, a\geq 0} H^*(\mathcal{O}_{\mathbf{g},a})[-\mu_{\mathbf{g},a}] \Rightarrow SH^*_+(Y)$$

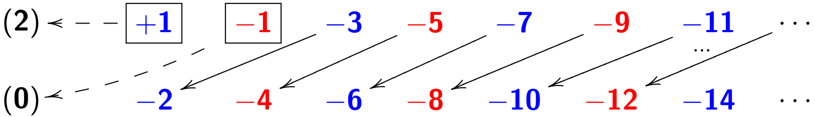
where $\mu_{\mathbf{g},a} \in \mathbb{Z}$ is a grading shift (*Conley-Zehnder index* of $\mathcal{O}_{\mathbf{g},a}$).
We believe the generators of $SH^*_+(Y)$ to be precisely the maxima x_g of Morse-Bott submfds $\mathcal{O}_{\mathbf{g},a}$ for $0 < a \leq 2\pi$ minimal for each $\mathbf{g} \in \mathrm{Conj}(G)$.
(Proved a slightly weaker statement).

Example: A_1-singularity $\mathbb{C}^2/\pm I$ and $Y = T^*\mathbb{CP}^1$

Slices $= \mathbb{RP}^3 = S^3/\pm I$ and $H^*(\mathbb{RP}^3) = \mathbb{K}[0] \oplus \mathbb{K}[-3]$ (for $\mathrm{char}(\mathbb{K}) = 0$)
Any initial point works, so

$$\mathcal{O}_{-I,\mathrm{odd}\cdot\pi} = \mathbb{RP}^3 \qquad \mathcal{O}_{+I,\mathrm{even}\cdot\pi} = \mathbb{RP}^3.$$

Morse-Bott spectral sequence $\oplus H^*(\mathcal{O}_{\mathbf{g},a})[-\mu_{\mathbf{g},a}] \Rightarrow SH^*_+(Y)$:



Explanation: **(0),(2)** $=$ Morse Complex of exceptional divisor $E = \mathbb{P}^1$.
O $=$ 1-orbits which lift from $\mathbb{C}^2/\pm I$ to $\mathbb{C}^2$ (Conj.Class $+\mathbf{I}$)
C $=$ 1-orbits which don't lift (Conj.Class $-\mathbf{I}$)
Thus $SH^*_+(Y)$ is generated by:

◇ $\boxed{+1}$ **half-great circle** of **age 1** in 1^{st} slice $\mathbb{RP}^3$, in $SH^1_+(Y) = H^2(Y)$.
For $(-I)^{-1} \in \mathrm{Aut}(\mathbb{C}^2)$ have evals $e^{\pi i}, e^{\pi i}$ so age $= 2\pi/2\pi = 1$.

◇ $\boxed{-1}$ **great circle** of **age 0** in 2^{nd} slice $\mathbb{RP}^3$, in $SH^{-1}_+(Y) = H^0(Y)$
Age grading: for $I^{-1} \in \mathrm{Aut}(\mathbb{C}^2)$ have evals e^{0i}, e^{0i} so age $= 0/2\pi = 0$.

S^1-Equivariant Symplectic Cohomology ESH^*

Want to avoid using $H^*(Y)$ in the argument [used it in the example].

Ordinary SH^*: is defined over the Novikov field $\mathbb{K}$. Think $\mathbb{C}((t))$.
S^1-Equivariant SH^*: over $\mathbb{K}[[u]]$-module $\mathbb{F} = \mathbb{K}((u))/u\mathbb{K}[[u]]$, $|u| = 2$.
Typical element: $k_p u^{-p} + \cdots + k_0 u^0$. Differential $\delta = \partial + u\delta_1 + u^2\delta_2 + \cdots$
Again $ESH^*(Y) = 0 \Rightarrow ESH^*_+(Y)[1] \cong EH^*(Y;\mathbb{K}) = H^*(Y;\mathbb{K}) \otimes \mathbb{F}$.

$\Rightarrow ESH^*_+(Y) = \oplus \mathbb{F}[-d_i]$ supported in degrees $d_i \in \{-1, 0, \ldots, 2n-2\}$.

Key: Now take $EH^*(\mathcal{O}_{\mathbf{g},a})$ not the ordinary $H^*(\mathcal{O}_{\mathbf{g},a})$.

EXAMPLE (continued): $\mathbb{C}^2/\pm I$. Each $\mathcal{O}_{\mathbf{g},a} = \mathbb{RP}^3$ contributes
$H^*_{S^1}(\mathcal{O}_{\mathbf{g},a}) = H^*(\mathbb{RP}^3/S^1) \cong H^*(S^3/S^1) \cong H^*(\mathbb{CP}^1) = \mathbb{K}[0] \oplus \mathbb{K}[-2]$
E_1-page of the spectral sequence $\oplus EH^*(\mathcal{O}_{\mathbf{g},a}) \Rightarrow ESH^*_+(Y)$:

$\boxed{+1}$	$\boxed{-1}$	-3	-5	-7	-9	-11	$\cdots$
-1	-3	-5	-7	-9	-11	-13	$\cdots$

Miracle: no differentials since all generators are in odd degrees!

General story: each $\mathcal{O}_{\mathbf{g},a}/S^1$ is a finite quotient of $\mathbb{CP}^k$ some k. For $\mathrm{char}(\mathbb{K}) = 0$, $EH^*(\mathcal{O}_{\mathbf{g},a}) \cong H^{*-1}(\mathbb{CP}^k)$ always in odd degrees, so:

$$\boxed{ESH^*_+(Y) = \oplus H^*(\mathbb{CP}^{k_{\mathbf{g},a}})[-1-\mu_{\mathbf{g},a}]}$$

The Gysin sequence relating SH^* and ESH^*

Example (continued): $\mathbb{C}^2/\pm I$, we find ESH^*_+ but need to recover SH^*_+.

$ESH^*_{+,-\mathrm{I}} = \mathbb{K}[-\mathbf{1}] \oplus \mathbb{K}[1] \oplus \mathbb{K}[3] \oplus \cdots = \mathbb{K}[-\mathbf{1}] \oplus \mathbb{K}[-\mathbf{1}]u^{-1} \oplus \cdots = \mathbb{F}[-\mathbf{1}]$

$ESH^*_{+,+\mathrm{I}} = \mathbb{K}[+\mathbf{1}] \oplus \mathbb{K}[3] \oplus \mathbb{K}[5] \oplus \cdots = \mathbb{K}[+\mathbf{1}] \oplus \mathbb{K}[+\mathbf{1}]u^{-1} \oplus \cdots = \mathbb{F}[+\mathbf{1}]$

The Symplectic Gysin sequence (Bourgeois-Oancea 2013):

$$\cdots \to SH^*_+(Y) \to ESH^*_+(Y) \xrightarrow{u} ESH^{*+2}_+(Y) \to SH^{*+1}_+(Y) \to \cdots$$

Remark. Classical Gysin sequence for S^1-bundle $\pi : E \to M$

$$\cdots \longrightarrow H_*(E) \xrightarrow{\pi_*} H_*(M) \xrightarrow{\cap e} H_{*-2}(M) \xrightarrow{\pi^{-1}} H_{*-1}(E) \to \cdots$$

which for $M = \mathcal{L}Y \times_{S^1} S^\infty$ (and $E = \mathcal{L}Y \times S^\infty \simeq \mathcal{L}Y$) becomes

$$\cdots \longrightarrow H_*(\mathcal{L}Y) \longrightarrow EH_*(\mathcal{L}Y) \longrightarrow EH_{*-2}(\mathcal{L}Y) \longrightarrow H_{*-1}(\mathcal{L}Y) \longrightarrow \cdots$$

Recall $ESH^*_+(Y) = \oplus \mathbb{F}[-d_i]$. Thus:

$$0 \to SH^{\mathrm{odd}}_+(Y) \to \oplus \mathbb{F}[-d_i] \xrightarrow{u} \oplus \mathbb{F}[-d_i] \to SH^{\mathrm{even}}_+(Y) \to 0$$

But $\mathbb{F} \xrightarrow{u} \mathbb{F}$ always surjective. So $SH^{\mathrm{even}}_+(Y) = 0$ (so $H^{\mathrm{odd}}(Y) = 0$) and $H^{2k}(Y) \cong SH^{2k-1}_+(Y) = \ker u$ of $\dim_{\mathbb{K}} = \mathrm{rk}_{\mathbb{F}}\, ESH^{2k-1}_+(Y) = \#\mathrm{Conj}_k(G)$ (the last equality is a non-trivial Conley-Zehnder index calculation).

In the example: $SH^*_+(Y) = \mathbb{K}[-\mathbf{1}] \oplus \mathbb{K}[+\mathbf{1}] = H^2(Y) \oplus H^0(Y)$.

Thank you for listening