Symplectic topology, Floer theory, and Fukaya categories

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A very brief survey of the research area.
December 2014.
A symplectic manifold \((M, \omega)\) is a smooth manifold with a closed non-degenerate 2-form \(\omega\).

- \(M = \mathbb{C}^n, \omega_0 = \sum dx_j \wedge dy_j\)
- \(M =\) orientable surface, \(\omega =\) area form
- Cotangent bundles \(T^*N, \omega = \sum dp_j \wedge dq_j = d(\sum p_j dq_j)\).
  
  In dynamical systems \(q = \) position, \(p = \) momentum.
- Kähler manifolds.

Key: all locally look like \((\mathbb{C}^n, \omega_0)\) \(\Rightarrow\) no local symplectic invariants.

\(\Rightarrow\) To tell them apart, need global invariants:

**Approach 1: use geometrical objects to distinguish them**

A submanifold \(L^n \subset M^{2n}\) is **Lagrangian** if \(\omega|_L = 0\).

- \(L = \mathbb{R}^n \subset \mathbb{C}^n\)
- Any embedded curve inside an orientable surface, e.g. \(S^1 \subset \mathbb{C}\)
- Clifford Torus \(S^1 \times \ldots \times S^1 \subset \mathbb{CP}^m\)
- \(L = \text{graph}(\alpha) \subset T^*N\) is Lagrangian \(\Leftrightarrow\) \(\alpha\) closed 1-form on \(N\)
- The torus in the sphere bundle of \(O(-k) \to \mathbb{CP}^m\) which projects to the Clifford torus
### The big picture

**Approach 2: construct cohomological invariants of symplectic mfds**

<table>
<thead>
<tr>
<th></th>
<th>$M$ closed</th>
<th>$M$ open or with $\partial M$</th>
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<tbody>
<tr>
<td>“closed strings”</td>
<td>$HF^<em>(H) \cong QH^</em>(M)$ Floer cohomology</td>
<td>$SH^*(M)$ Symplectic cohomology</td>
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<tr>
<td>“open strings”</td>
<td>$HF^*(L_1, L_2)$ Lagrangian Floer c.</td>
<td>$HW^*(L_1, L_2)$ Wrapped Floer c.</td>
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**Fukaya category**: loosely, package all Lagrangians $L \subset M$ up into a category, using $HF^*(L_1, L_2)$ as morphism spaces.

Lots of algebraic structure: $HF^*(L_1, L_2)$ are $QH^*(M)$-modules.

**Mirror symmetry conjecture** (Kontsevich ’94): There are mirror pairs, $(M, \omega)$ symplectic manifold and $(X, J)$ complex variety,

- Studying Lagrangian submanifolds $L \subset M$ ↔ Studying holomorphic vector bundles $V \to X$

\[
\begin{align*}
\text{Fukaya category} & \quad \mathcal{F}(M) \text{ of } M \\
\text{Equivalence on the derived categories} & \quad \text{Category of Coherent Sheaves on } X
\end{align*}
\]

Often $SH^*(M) \cong HH_*(\text{wrapped } \mathcal{F}(M)) \cong HH_*(D^b\text{Coh}(X))$. 

What is Floer cohomology?

$\mathcal{M}$ Symplectic Manifold

$H : \mathcal{M} \to \mathbb{R} \quad dH = \omega(\cdot, X_H)$

1-periodic Hamiltonian orbits generate $CF^\ast(H)$

Closed Strings

Product counts $u : 3$-punctured sphere $\to \mathcal{M}$

$(du - X_H \otimes \beta)^{0,1} = 0$

Differential counts $u : \mathbb{R} \times S^1 \to \mathcal{M}$

$\partial_s u + J \partial_t u = -\nabla H$ (elliptic PDE)

Floer cohomology is formally the Morse cohomology of an action functional on the free loop space of $\mathcal{M}$. 
What is Floer cohomology?

\((M, \omega)\) Symplectic Manifold

\[ H : M \to \mathbb{R} \quad dH = \omega(\cdot, X_H) \]

1-periodic Hamiltonian

orbits generate

\( CF^*(H) \)

Critical point of \( H \)

Differential counts

\( u : \mathbb{R} \to M \)

\[ \partial_s u = -\nabla H \]

\( u : \mathbb{R} \times S^1 \to M \)

\[ \partial_s u + J\partial_t u = -\nabla H \]

\[ \Rightarrow HF^*(H) \cong \text{Morse cohomology} \cong H^*(M) \text{ as a vector space} \]

\[ \Rightarrow HF^*(H) \cong \text{Quantum cohomology} \ QH^*(M) \text{ as a ring} \]
The choice of $H$

**Floer ’87:** Closed $M \Rightarrow H$ does not matter: $HF^*(H) \cong QH^*(M)$.
Example: $QH^*(\mathbb{P}^m) = \Lambda[\omega]/(\omega^{1+m} - t)$, where $\Lambda = \mathbb{Z}((t))$.

**For non-compact (and convex) $M$: growth of $H$ matters**
Example: $T^* S^1$: can ensure Hamiltonian flow = geodesic flow

Winding 1
Winding 2
Winding 3

Increase slope of $H$ at $\infty \Rightarrow$ new 1-periodic geodesics.

**Symplectic cohomology:**
$SH^*(T^* S^1) = \lim HF^*(H) = \bigoplus_{\mathbb{Z}} H^*(S^1) \cong H_*(\mathcal{L}S^1)$

**Viterbo ’94:**
$SH^*(T^* N; \mathbb{Z}/2) \cong H_{n-*}(\mathcal{L}N; \mathbb{Z}/2)$
(Free loop space: $\mathcal{L}N = C^\infty(S^1, N)$)

Also proved by **Abbondandolo-Schwarz**, and **Salamon-Weber**.

In general: there is a map $QH^*(M) \to SH^*(M)$, however $SH^*(M)$ is usually zero or infinite dimensional.
If $SH^*(M) \neq 0$ you usually don’t stand a chance at calculating it.
What is the Fukaya category?

$(M, \omega)$ Symplectic Manifold
$H : M \to \mathbb{R} \quad dH = \omega(\cdot, X_H)$

Open Strings with Lagrangian boundary conditions

Flow $L_0$

(Change $L_0, J$)

Intersections $L_0 \cap L_1$

generate $CF^*(L_0, L_1)$

Hamiltonian orbits

$u : \mathbb{R} \times [0, 1] \to M$
$
\partial_s u + J\partial_t u = -\nabla H$

Differential counts holomorphic strips

$\partial_s u + J\partial_t u = 0$
What is the Fukaya category?

\[(M, \omega)\] Symplectic Manifold

- Intersections \(L_0 \cap L_1\) generate \(CF^*(L_0, L_1)\)
- Differential counts holomorphic strips \(\partial_s u + J \partial_t u = 0\)
- Product counts holomorphic triangles
- \(A_{\infty} - \text{category}\)
- Fukaya category \(F(M)\)
- Objects: Lagrangians
- Morphisms: \(CF^*(L_0, L_1)\)
- Not Associative
- Higher operations count holomorphic polygons
Dependence on $H$:

Example: $L = \text{zero section } S^1 \subset T^*S^1$:

For non-compact (and convex) $M$: growth of $H$ matters

Wrapped Fukaya category $W(M)$:
Obj: also non-compact Lagrangians (Legendrian at $\infty$).
Morphs: $CW^*(L_0, L_1)$ generated by Hamiltonian orbits

Example: $L = \text{fibre } \subset T^*S^1$:

Wrapped Floer cohomology:
$$HW^*(L, L) = \lim_{-\to} HF^*(\text{flow}_H(L), L) = \bigoplus \mathbb{Z}$$
$$\cong H_*(\Omega S^1) \text{ (based loop space)}$$

Abbondandolo-Schwarz '05: for $L = \text{fibre } \subset T^*N$,
$$HW^*(L, L; \mathbb{Z}/2) \cong H_{n-*}(\Omega N; \mathbb{Z}/2)$$
Relating the Floer cohomology to the Fukaya category

Open-Closed string map

\[ CF^*(L_4, L_0) \otimes CF^*(L_3, L_4) \otimes \ldots \otimes CF^*(L_0, L_1) \rightarrow HF^*(H) \cong QH^*(M) \]

\[ \Rightarrow OC : HH^*(F(M)) \rightarrow QH^*(M) \text{ on Hochschild homology} \]

Similarly can construct:

\[ OC : HH^*(W(M)) \rightarrow SH^*(M) \]

(Abouzaid 2010 in exact case
Ritter & Smith 2012 in monotone case)