# Symplectic topology, Floer theory, and Fukaya categories

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A very brief survey of the research area.
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# The big picture

A **symplectic manifold** $(M, \omega)$ is a smooth manifold with a closed non-degenerate 2-form $\omega$.

- $M = \mathbb{C}^n$, $\omega_0 = \sum dx_j \wedge dy_j$
- $M =$ orientable surface, $\omega =$ area form
- Cotangent bundles $T^*N$, $\omega = \sum dp_j \wedge dq_j = d(\sum p_j dq_j)$.
  In dynamical systems $q =$ position, $p =$ momentum.
- Kähler manifolds.

Key: all locally look like $(\mathbb{C}^n, \omega_0) \Rightarrow$ no *local* symplectic invariants.

$\Rightarrow$ To tell them apart, need global invariants:

*Approach 1: use geometrical objects to distinguish them*

A submanifold $L^n \subset M^{2n}$ is **Lagrangian** if $\omega|_L = 0$.

- $L = \mathbb{R}^n \subset \mathbb{C}^n$
- Any embedded curve inside an orientable surface, e.g. $S^1 \subset \mathbb{C}$
- Clifford Torus $S^1 \times \cdots \times S^1 \subset \mathbb{CP}^m$
- $L = \mathrm{graph}(\alpha) \subset T^*N$ is Lagrangian $\Leftrightarrow$ $\alpha$ closed 1-form on $N$
- The torus in the sphere bundle of $\mathcal{O}(-k) \to \mathbb{CP}^m$ which projects to the Clifford torus

# The big picture

*Approach 2: construct cohomological invariants of symplectic mfds*

| | $M$ **closed** | $M$ **open or with** $\partial M$ |
|---|---|---|
| **"closed strings"** | $HF^*(H) \cong QH^*(M)$ Floer cohomology | $SH^*(M)$ Symplectic cohomology |
| **"open strings"** | $HF^*(L_1, L_2)$ Lagrangian Floer c. | $HW^*(L_1, L_2)$ Wrapped Floer c. |

**Fukaya category**: loosely, package all Lagrangians $L \subset M$ up into a category, using $HF^*(L_1, L_2)$ as morphism spaces.

*Lots of algebraic structure: $HF^*(L_1, L_2)$ are $QH^*(M)$-modules.*

**Mirror symmetry conjecture** (Kontsevich '94): There are mirror pairs, $(M, \omega)$ symplectic manifold and $(X, J)$ complex variety,

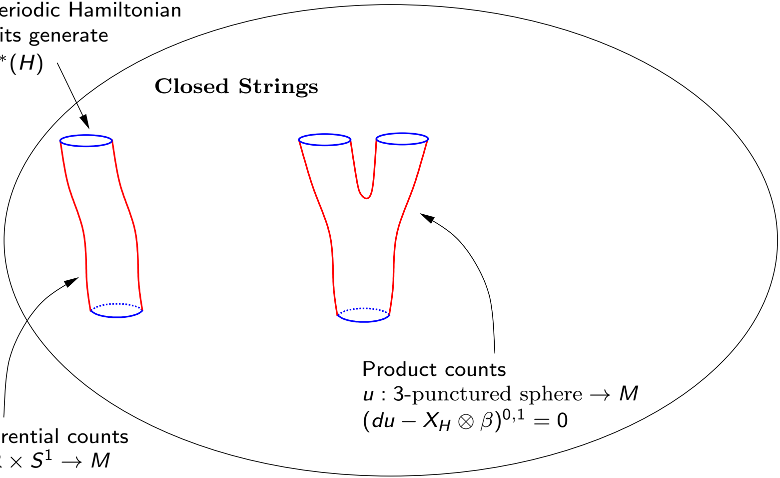
| | | |
|---|---|---|
| Studying Lagrangian submanifolds $L \subset M$ | $\longleftrightarrow$ | Studying holomorphic vector bundles $V \to X$ |
| Fukaya category $\mathcal{F}(M)$ of $M$ | Equivalence on the $\longleftrightarrow$ derived categories | Category of Coherent Sheaves on $X$ |

Often $SH^*(M) \cong \mathrm{HH}_*(\text{wrapped } \mathcal{F}(M)) \cong \mathrm{HH}_*(D^b\mathrm{Coh}(X))$.

# What is Floer cohomology?

$(M, \omega)$ **Symplectic Manifold**
$H : M \to \mathbb{R} \qquad dH = \omega(\cdot, X_H)$

1-periodic Hamiltonian orbits generate $CF^*(H)$

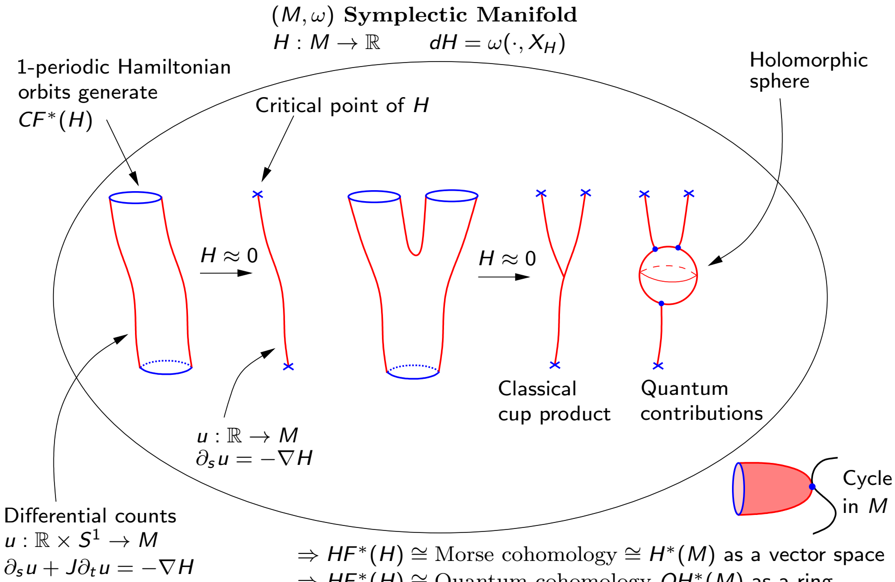



Product counts
$u$ : 3-punctured sphere $\to M$
$(du - X_H \otimes \beta)^{0,1} = 0$

Differential counts
$u : \mathbb{R} \times S^1 \to M$
$\partial_s u + J\partial_t u = -\nabla H$ (elliptic PDE)

Floer cohomology is formally the Morse cohomology of an action functional on the free loop space of $M$.

# What is Floer cohomology?



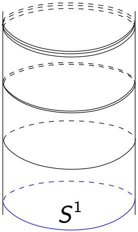

$\Rightarrow HF^*(H) \cong$ Morse cohomology $\cong H^*(M)$ as a vector space

$\Rightarrow HF^*(H) \cong$ Quantum cohomology $QH^*(M)$ as a ring

# The choice of $H$

**Floer '87:** Closed $M \Rightarrow H$ does not matter: $HF^*(H) \cong QH^*(M)$.
Example: $QH^*(\mathbb{P}^m) = \Lambda[\omega]/(\omega^{1+m} - t)$, where $\Lambda = \mathbb{Z}((t))$.

**For non-compact (and convex) M: growth of $H$ matters**
Example: $T^*S^1$: can ensure Hamiltonian flow = geodesic flow

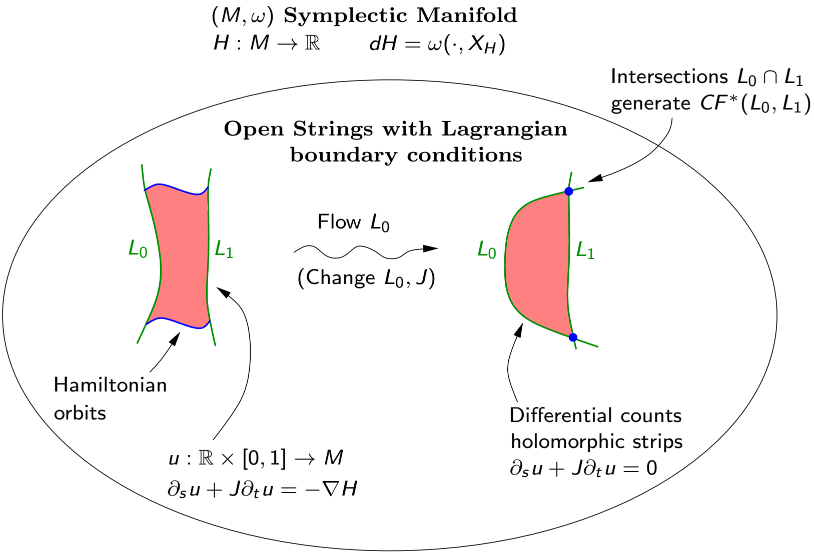



Increase slope of $H$ at $\infty \Rightarrow$ new 1-periodic geodesics.
**Symplectic cohomology:**
$SH^*(T^*S^1) = \varinjlim HF^*(H) = \bigoplus_{\mathbb{Z}} H^*(S^1) \cong H_*(\mathcal{L}S^1)$

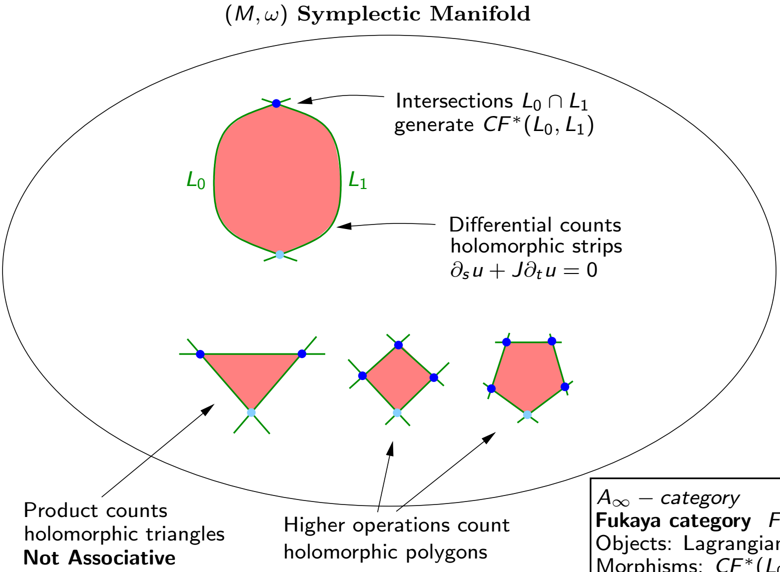
**Viterbo '94:** $\boxed{SH^*(T^*N; \mathbb{Z}/2) \cong H_{n-*}(\mathcal{L}N; \mathbb{Z}/2)}$
(Free loop space: $\mathcal{L}N = C^\infty(S^1, N)$)
Also proved by **Abbondandolo-Schwarz**, and **Salamon-Weber**.

In general: there is a map $QH^*(M) \to SH^*(M)$, however $SH^*(M)$ is usually zero or infinite dimensional.
If $SH^*(M) \neq 0$ you usually don't stand a chance at calculating it.

# What is the Fukaya category?



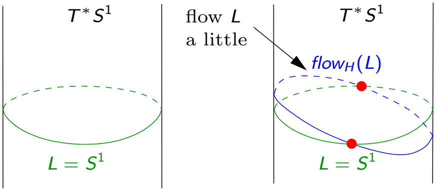
# What is the Fukaya category?




*$A_\infty$ − category*
**Fukaya category** $F(M)$
Objects: Lagrangians
Morphisms: $CF^*(L_0, L_1)$

# Dependence on $H$:

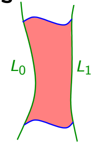
Example: $L = \text{zero section } S^1 \subset T^*S^1$:



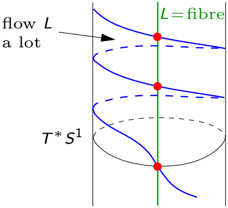

Lagrangian Floer cohomology:
$HF^*(L,L) = \mathbb{Z} \oplus \mathbb{Z} \cong H_*(L)$

**For non-compact (and convex) M: growth of $H$ matters**

*Wrapped Fukaya category $W(M)$:*
Obj: also non-compact Lagrangians (Legendrian at $\infty$).
Morphs: $CW^*(L_0, L_1)$ generated by Hamiltonian orbits




Example: $L = \text{fibre} \subset T^*S^1$:




Wrapped Floer cohomology:

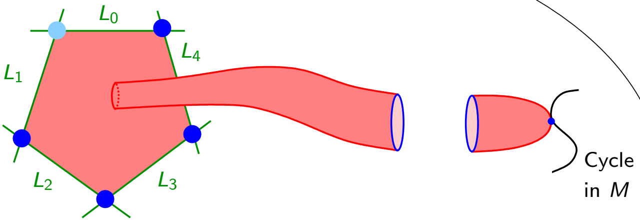
$$HW^*(L,L) = \varinjlim HF^*(\text{flow}_H(L), L) = \bigoplus_{\mathbb{Z}} \mathbb{Z}$$
$$\cong H_*(\Omega S^1) \text{ (based loop space)}$$

**Abbondandolo-Schwarz '05:** for $L = \text{fibre} \subset T^*N$,

$$HW^*(L,L;\mathbb{Z}/2) \cong H_{n-*}(\Omega N;\mathbb{Z}/2)$$

# Relating the Floer cohomology to the Fukaya category

**Open-Closed string map**




$$\underline{CF^*(L_4, L_0)} \otimes CF^*(L_3, L_4) \otimes \dots \otimes CF^*(L_0, L_1) \to HF^*(H) \cong QH^*(M)$$

$\Rightarrow$ $\boxed{\mathcal{OC} : \mathrm{HH}_*(F(M)) \to QH^*(M)}$ on Hochschild homology

Similarly can construct:

$\boxed{\mathcal{OC} : \mathrm{HH}_*(W(M)) \to SH^*(M)}$

**(Abouzaid 2010 in exact case**
**Ritter & Smith 2012 in monotone case)**