# Equivariant Floer theory for symplectic $\mathbb{C}^*$-manifolds


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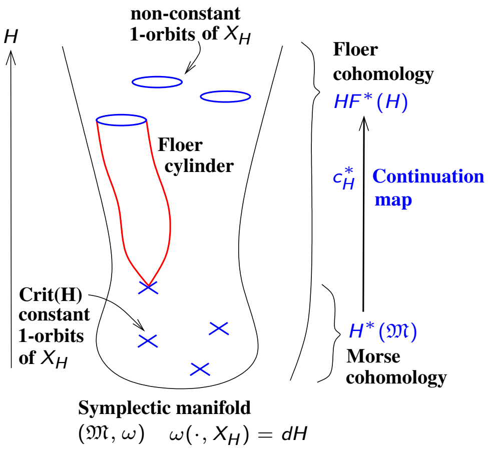
(I thank Stanford University for their hospitality during my sabbatical visit 2022-2023 during which parts of this research took place)



Joint work with Filip Živanović
Simons Center for Geometry and Physics


Symplectic Zoominar
CRM-Montréal, Princeton/IAS, Tel Aviv, and Paris
9 May 2025.

# Idea 1: using Floer theory to describe ordinary cohomology



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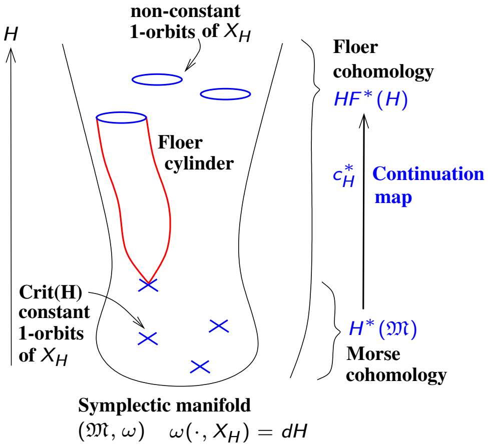


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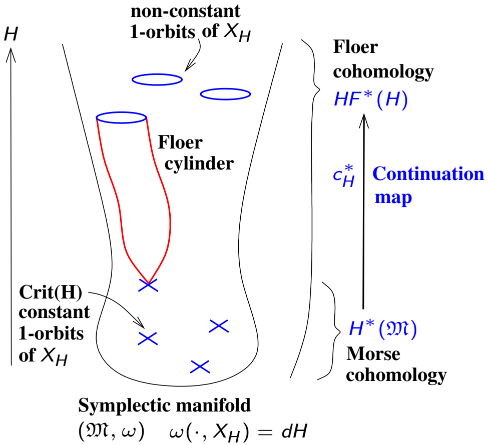


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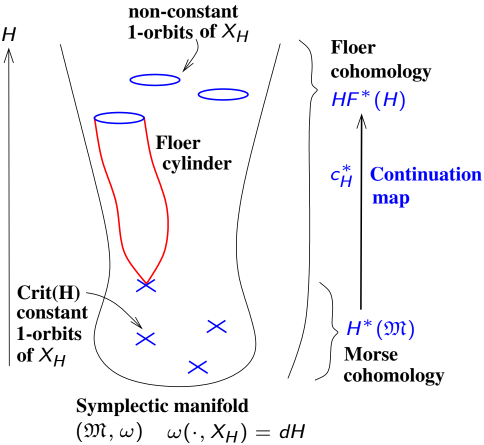


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**Example**: CY resolution of singularities $\mathfrak{M} \to \mathbb{C}^n/G$ for finite $G \leq SL(n, \mathbb{C})$
$H^*(\mathfrak{M})$ has basis labelled by Conjugacy Classes $(G)$ (McKay Correspondence)
McLean-R.'23 $SH_+^{*-1}(\mathfrak{M}) \cong H^*(\mathfrak{M})$ for isolated singularities $\mathbb{C}^n/G$
and non-constant orbits are naturally labelled by Conjugacy Classes $(G)$.

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**Typical setup:** family of Hamiltonians $H_a$, $a \in [0, \infty) \setminus (\mathrm{discrete})$, the $H_a$ grow at $\infty$ as $a$ increases, and $c^*$-maps are *compatible* for $a \leq b$:

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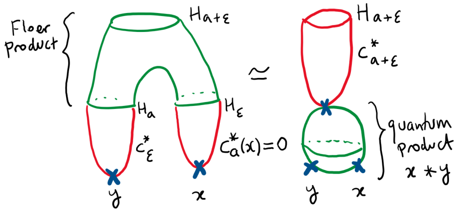
$$\begin{array}{ccc} QH^* \otimes QH^* & \xrightarrow{c_\varepsilon^* \otimes c_a^*} & HF^*(H_\varepsilon) \otimes HF^*(H_a) \\ \downarrow & & \downarrow \\ QH^* & \xrightarrow{c^*_{a+\varepsilon}} & HF^*(H_{a+\varepsilon}) \end{array}$$



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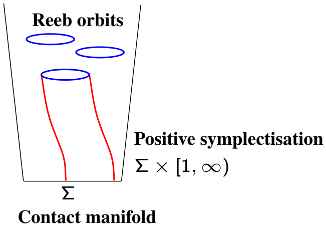


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$\diamond$ $\exists$ Hamiltonian $S^1$-action which extends to $I$-holomorphic $\mathbb{C}^*$-action;
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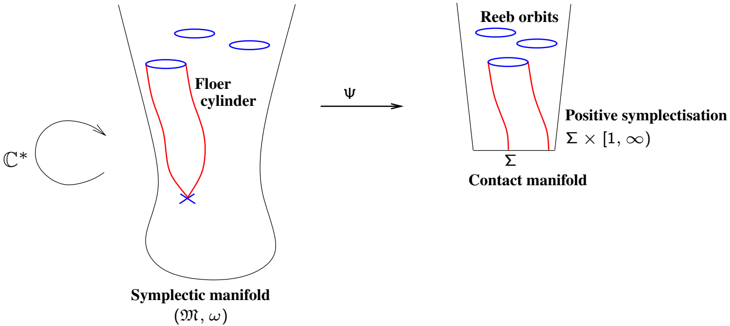


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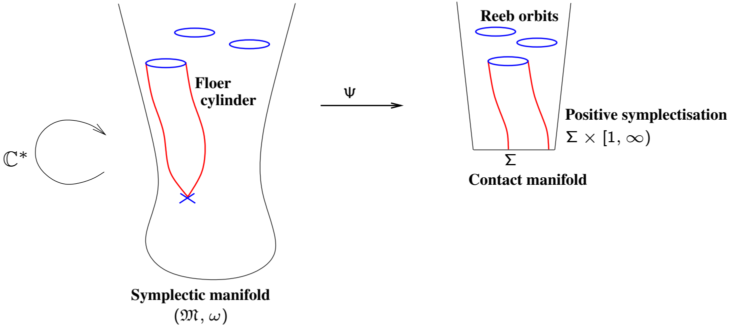


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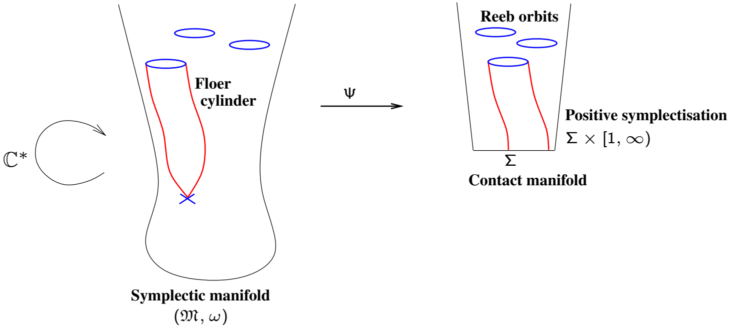
**Basic example:** $\mathbb{C}^2/(\mathbb{Z}/3) \hookrightarrow \mathbb{V}(XY - Z^3) \subset \mathbb{C}^3$, $(x,y) \mapsto (x^3, y^3, xy)$, $\mathbb{C}^*$-action $(tx, ty)$. Blow-up 0 get $A_2$-resolution $\mathfrak{M}$, $\Psi = (X^2, Y^2, Z^3) \in \mathbb{C}^3$.

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$\exists$ **Many examples of symplectic $\mathbb{C}^*$-manifolds, often hyperkähler**:

| | |
|---|---|
| $T^*\mathbb{C}P^n$, $T^*(\text{Flag Variety})$ | Quiver varieties |
| Negative complex vector bundles | Semiprojective toric varieties |
| CY Resolutions of singularities $\mathbb{C}^n/G$ | Hypertoric manifolds |
| Conical Symplectic Resolutions | Moduli spaces of Higgs bundles |
| Springer resolutions of Slodowy varieties | $\mathbb{C}^*$-equivariant projective morphisms |

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Let $H =$ moment map of $S^1$-action on symplectic $\mathbb{C}^*$-manifold $\mathfrak{M}$.

**Theorem (RŽ'23)**

*1) Can do Floer theory for Hamiltonians $H_k$ that are $kH$+constant at $\infty$.*
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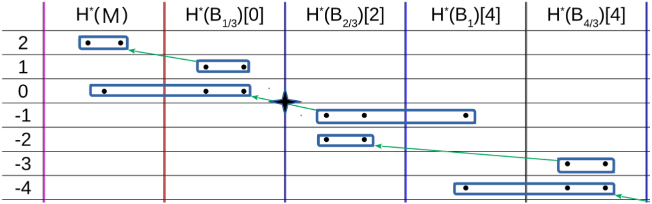

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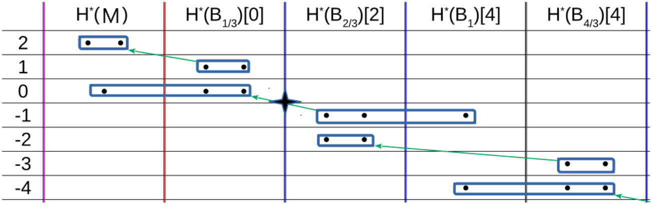

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**Examples:**
**CY** ($c_1(\mathfrak{M}) = 0$): $SH^*(\mathfrak{M}) = 0$, $\ker c_k^* = QH^*(\mathfrak{M}) \cong SH_+^{*-1}(\mathfrak{M})$ for $k \gg 0$.
**Fano** $\mathcal{O}(-1) \to \mathbb{C}P^n$ has $\dim QH^* = n+1$, $\dim E_0 = 1$, $\dim SH^* = n$.

# $S^1$-equivariant cohomology: some classical recollections

$\exists$ three versions, depending on what you want it to be for a point $\bullet$

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**Morse picture**

$x = \min f, \ y = \max f$

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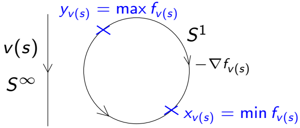


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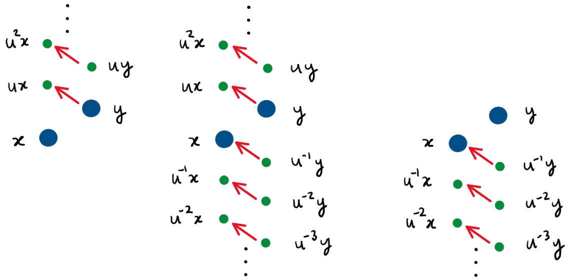


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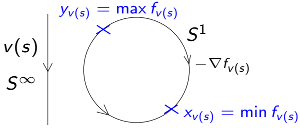


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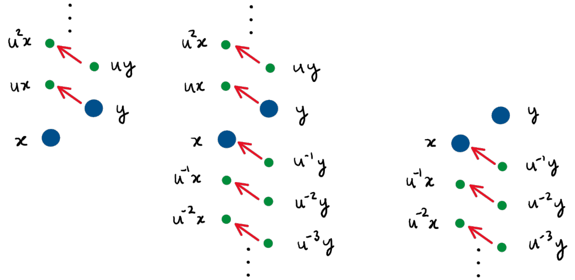


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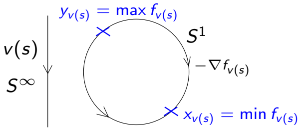


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**Adapting Kirwan '84 to our $\mathfrak{M}$, "Equivariant formality":**
$\exists$ non-canonical iso $E^{\diamondsuit}H^*(\mathfrak{M}) \cong H^*(\mathfrak{M}) \otimes E^{\diamondsuit}H^*(\bullet)$ of $\mathbb{K}[\![u]\!]$-modules

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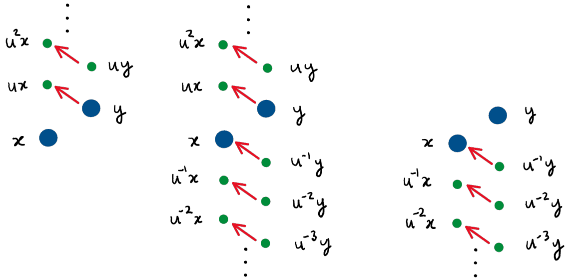


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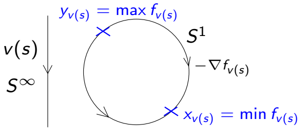


**Borel model** $S^\infty \times_{S^1} S^1$

**Adapting Kirwan '84 to our $\mathfrak{M}$, "Equivariant formality":**

$\exists$ non-canonical iso $E^\diamondsuit H^*(\mathfrak{M}) \cong H^*(\mathfrak{M}) \otimes E^\diamondsuit H^*(\bullet)$ of $\mathbb{K}[\![u]\!]$-modules

**Borel's localisation theorem:** For $\mathrm{char}\,\mathbb{K} = 0$,

$E^-H^*(\text{Compact mfd}) \to E^-H^*(\text{Fixed Locus})$ is an iso after $u$-localisation.

Equivalently: $E^\infty H^*(\text{Compact mfd}) \cong H^*(\text{Fixed Locus})(\!(u)\!)$.

Two $S^1$-actions on loops $x(t) \subset \mathfrak{M}$, $\mathbb{C}^*$-action $\varphi$ and loop-reparametrisation:

$$\theta \cdot x(t) = \varphi_{a\theta}(x(t - b\theta)) \qquad \textbf{"weight (a,b)" action}$$

## $S^1$-equivariant Floer theory: Localisation theorem

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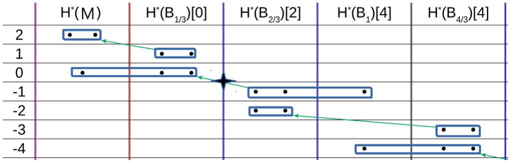

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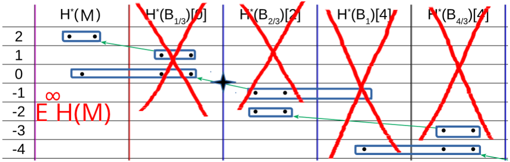

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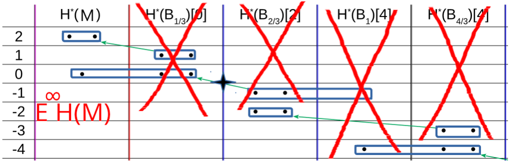


Zhao'19 Localisation theorem for $\mathfrak{M}$ Liouville, weight (0,1), different proof.

## Filtrations on $EH^*(\mathfrak{M})$ via Floer theory

For $E^+H^*(\mathfrak{M})$: filtration $\boxed{\mathscr{E}_k \;:=\; \ker(c_k^*)}$ $\mathbb{K}(\!(u)\!)/u\mathbb{K}[\![u]\!]$-mod

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$\exists$ LES: $\cdots \to E^-HF \to E^\infty HF \to E^+HF \to \cdots$ respecting filtrations, which becomes a SES in the limit: $0 \to E^-SH \to E^\infty SH \to E^+SH \to 0$

## Computing $E^{\diamondsuit} SH^* = \varinjlim \left( E^{\diamondsuit} QH^* \to E^{\diamondsuit} HF^*(H_1) \to E^{\diamondsuit} HF^*(H_2) \to \cdots \right)$

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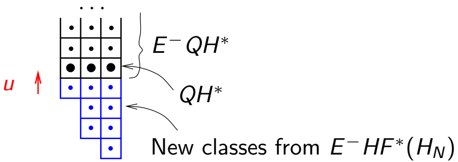

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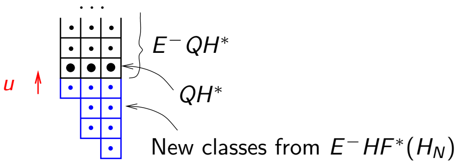


**Non-trivial example:** $\mathfrak{M} = T^*\mathbb{C}P^1$, for $N = 1$:

Might expect "naive inclusion"
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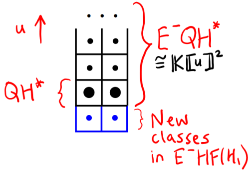


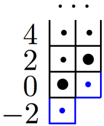

Graded picture

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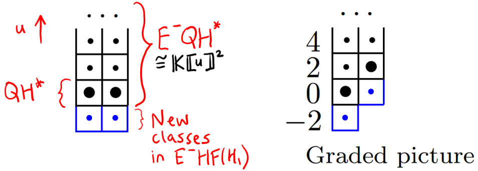
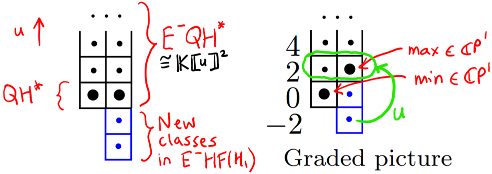



But actually, this happens:




what this implies about the filtration: $\dim_{\mathbb{K}}(\mathscr{F}_0^0 \cap \mathscr{F}_1^2) = 1$.

Recall: $E_0 = \ker \mathcal{Q}^{N \gg 0} \subset QH^*(\mathfrak{M})$, and that $SH^*(\mathfrak{M}) \cong QH^*(\mathfrak{M})/E_0$.
Non-canonically: $QH^*(\mathfrak{M}) \cong SH^*(\mathfrak{M}) \oplus E_0$, as $\mathbb{K}$-modules.

# Equivariant formality: computation in the CY / Fano cases

Recall: $E_0 = \ker \mathcal{Q}^{N \gg 0} \subset QH^*(\mathfrak{M})$, and that $SH^*(\mathfrak{M}) \cong QH^*(\mathfrak{M})/E_0$.
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*If $c_1(\mathfrak{M}) \in \mathbb{R}_{\geq 0}[\omega]$, then $\exists$ non-canonical $\mathbb{K}[\![u]\!]$-module isomorphisms:*
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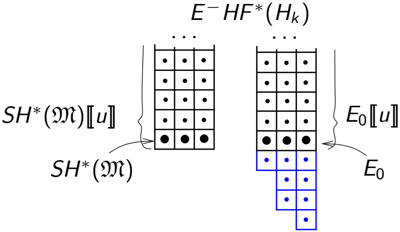

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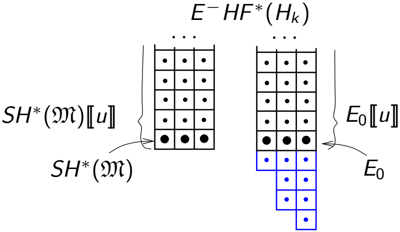
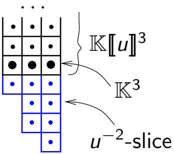


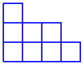

**CY example.** $T^*\mathbb{C}P^1 = \mathcal{O}_{\mathbb{C}P^1}(-2)$: $E^- SH^* \cong \mathbb{K}(\!(u)\!)^2$.
**Fano example.** $\mathrm{Bl}_0(\mathbb{C}^2) = \mathcal{O}_{\mathbb{C}P^1}(-1)$: $E^- SH^* \cong \mathbb{K}[\![u]\!] \oplus \mathbb{K}(\!(u)\!)$.

# The filtration polynomial

**Basic example:** $\begin{pmatrix} u & 0 & 0 \\ 0 & u^3 & 0 \\ 0 & 0 & u^4 \end{pmatrix}^{-1} \cdot \mathbb{K}[\![u]\!]^3 = u^{-1}\mathbb{K}[\![u]\!] \oplus u^{-3}\mathbb{K}[\![u]\!] \oplus u^{-4}\mathbb{K}[\![u]\!]$



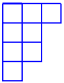

Young diagram for the $j_i$



$j_1 = 1$
$j_2 = 3$
$j_3 = 4$

(French notation: row lengths non-decreasing)

Dual Young diagram: the $d_j$



$d_1 = 3$
$d_2 = 2$
$d_3 = 2$
$d_4 = 1$

$(d_0 = r = 3, d_{\geq 5} = 0)$

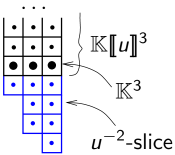
$d_n = \dim_{\mathbb{K}}(u^{-n}\text{-slice})$, and $u^{j_1}, u^{j_2}, u^{j_3}$ are the invariant factors $u, u^3, u^4$.

(RŽ'24) **Filtration polynomial**:

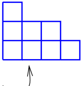
$h := d_0 + d_1 t + \cdots + d_j t^j + \cdots$ where $d_j := \#\{\text{invariant factors } u^{\geq j}\}$.

In the basic example: $h = 3 + 3t + 2t^2 + 2t^3 + t^4$.

# The filtration polynomial



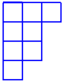

Young diagram for the $j_i$



$j_1 = 1$
$j_2 = 3$
$j_3 = 4$

(French notation: row lengths non-decreasing)

Dual Young diagram: the $d_j$



$d_1 = 3$
$d_2 = 2$
$d_3 = 2$
$d_4 = 1$

$(d_0 = r = 3, d_{\geq 5} = 0)$

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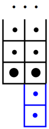
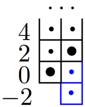
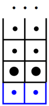
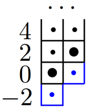
$\mathbb{K}[\![u]\!]$ is a discrete valuation ring with parameter $u$, so the Smith normal form of $(c_N^*)^{-1}$ is encoded by invariant factors that are powers of $u$:






Graded picture






"naive inclusion"

Graded picture

Invariant factors $(1, u^2)$
Filtration polynomial $h = 2 + t + t^2$

Invariant factors $(u, u)$
Filtration polynomial $h = 2 + 2t$

# Filtration polynomial determines a Hilbert-Poincaré series

$$(c_k^*)^{-1}(E^- HF^*(H_k)) \subset E^- QH^*(\mathfrak{M}) \otimes \mathbb{K}((u)) \qquad \text{for slopes } k \in [0,\infty)$$

$\Rightarrow$ nested **Young diagrams** as increase $k$, **filtration polys** $h_k(t) = \sum d_j(k)\, t^j$

**Theorem.** (RŽ'24) $d_j(k) = \dim_{\mathbb{K}} \mathscr{F}_k^j / u\mathscr{F}_k^{j-1}$ $(d_j(k) \leq d_j(k')$ for $k \leq k')$

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**Examples.** $\mathfrak{M} = \mathbb{C}$: $h_k = 1 + t + \cdots + t^k$ and $P_k = t^k$

$\mathfrak{M} = \mathrm{Bl}_0(\mathbb{C}^2) = \mathcal{O}_{\mathbb{C}P^1}(-1)$: $h_k = 2 + t + \cdots + t^k$ and $P_k = 1 + t^k$

$\mathfrak{M} = T^*\mathbb{C}P^1 = \mathcal{O}_{\mathbb{C}P^1}(-2)$: $h_k = 2 + 2t + \cdots + 2t^{k-1} + \diamond$,

where $\diamond = t^k + t^{k+1}$ or $2t^k$. So $P_k = t^{k-1} + t^{k+1}$ or $2t^k$.

# Filtration polynomial determines a Hilbert-Poincaré series

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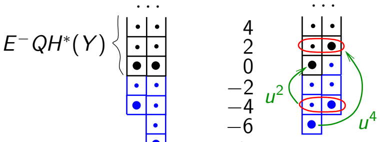
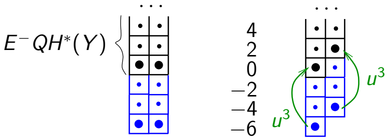



$k = 3$ case — Graded picture

$h_3 = N_3 = 2 + 2t + 2t^2 + t^3 + t^4$ possibility



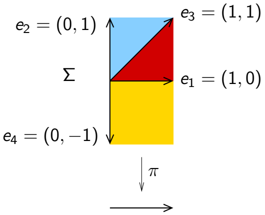

Graded picture

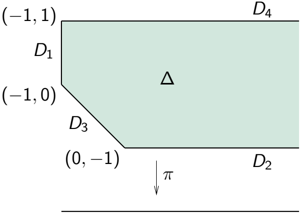
$h_3 = Z_3 = 2 + 2t + 2t^2 + 2t^3$ possibility

Natural class of non-compact toric mfds: GIT quotients of a cx torus action on a cx vector space (Hausel-Sturmfels'02). Toric mfds $\mathfrak{M}$ have lots of $\mathbb{C}^*$-actions $\varphi_v$ from $v \in N = \mathrm{Hom}(\mathbb{C}^*, \mathbb{T})$, as $\mathfrak{M} = \mathrm{Closure}(\mathbb{T})$.

There is a combinatorial description of $\mathfrak{M}$ in terms of a fan $\Sigma$.
Moment polytope $\Delta = \mathrm{image}(\mu : \mathfrak{M} \to \mathbb{R}^n) = \{x \in \mathbb{R}^n : \langle x, e_i \rangle \geq \lambda_i\}$,
inward normals $e_i$, toric divisors $D_i = \mu^{-1}(\{x \in \mathbb{R}^n : \langle x, e_i \rangle = \lambda_i\}) \subset \mathfrak{M}$.
Landau-Ginzburg superpotential $W = \sum T^{-\lambda_i} z^{e_i}$.
Jacobian ring $\mathrm{Jac}(W) = \mathbb{K}[z_i^{\pm 1}]/(\partial_{z_i} W)$.

**Example:**
**Fano SP-toric surface** $\pi : \mathrm{Bl}_{\{0\}\times\{[1:0]\}}(\mathbb{C}\times\mathbb{P}^1) \to \mathbb{C}$. Holo curves appear at infinity: $\pi^{-1}(c) \cong \mathbb{P}^1$ for $c \neq 0 \in \mathbb{C}$. Fan $\Sigma$, moment polytope $\Delta$ are:



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**Theorem (RŽ'23 (and in RŽ'25 the equivariant version of this theorem) )**

*Let $\mathfrak{M}$ be Fano SP-toric. Any $v \in N \cap \mathrm{Int}(|\Sigma|) \Rightarrow (\mathfrak{M}, \varphi_v)$ symplectic $\mathbb{C}^*$-manifold, with* **rotation element** $\mathcal{Q}_{\varphi_v} = x^v := x_{j_1}^{v_1} x_{j_2}^{v_2} \cdots x_{j_n}^{v_n}$, *where $v = v_1 e_{j_1} + \cdots + v_n e_{j_n}$ in the basis $e_{j_a}$ of some maximal cone, and*

$$\mathbb{K}[x_1, \ldots, x_r]/\mathcal{J} \cong QH^*(\mathfrak{M}), \; x_i \mapsto \mathrm{PD}[D_i],$$

$$QH^*(\mathfrak{M})[x^{\pm v}] \cong \mathbb{K}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]/\mathcal{J} \cong SH^*(\mathfrak{M}, \varphi_v) \cong \mathrm{Jac}(W)$$

$$x_i \mapsto c^*(\mathrm{PD}[D_i]) \mapsto T^{-\lambda_i} z^{e_i},$$

*$\mathcal{J}$ := ideal generated by linear & quantum Stanley-Reisner relations.*

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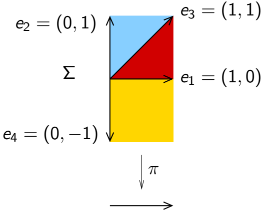


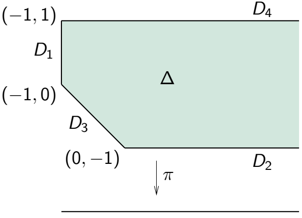


Pick any $v \in \mathbb{Z}_{>0} \times \mathbb{Z} \subset N = \mathbb{Z}^2$, then for $\mathbb{C}^*$-action $\varphi_v$:

$$\begin{aligned}
H^*(\mathfrak{M};\mathbb{Z}) &\cong \mathbb{Z}[x_1,x_2]/(x_1x_2,\ x_1^2,\ x_2^2),\\
QH^*(\mathfrak{M}) &\cong \mathbb{K}[x_1,x_2]/(x_1x_2+Tx_1,\ x_1^2,\ x_2^2+Tx_1-T^2),\\
SH^*(\mathfrak{M},\varphi_v) &\cong QH^*(Y)[x^{\pm v}]\\
&\cong \mathbb{K}[x_1^{\pm1},x_2^{\pm1}]/(x_1x_2+Tx_1,\ x_1^2,\ x_2^2+Tx_1-T^2)\\
&= 0,\\
E_0 &= QH^*(\mathfrak{M}),\\
E^-SH^*(\mathfrak{M}) &\cong E_0((u)) \cong QH^*(\mathfrak{M})((u)) \cong E^\infty SH^*(\mathfrak{M}).
\end{aligned}$$

Thank you for listening!