The development of mathematical theories. A few remarks.

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Résumé

In this talk, we shall argue that the natural tendency of mathematical theories is to spread and ramify until they reach the foundations of mathematics. This is paradoxical, because when the foundations of mathematics are altered, so is the language in which in the theory was originally formulated and it is then not clear anymore what the theory *said* to begin with.

The main thesis that I shall try to defend in this talk is that mathematical theories have an inherent tendency to spread as far as the very foundations of mathematics. I shall first give a few examples of this phenomenon and I after that I shall address the paradoxical nature of this process.

All the material presented here is very much work in progress and a lot more research would be required to see whether my thesis really holds water. I apologise in advance for my insufficient knowledge of the history of mathematics.

1 First example. Arithmetics and Geometry.

Let me first recall a few well-known facts about the history of Arithmetics and Geometry.

• Arithmetics and Geometry appear first in their modern guise in Euclid's Elements. The Elements introduce a completely new approach to these two subjects : the axiomatic

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method. The idea of the axiomatic method is to introduce a few basic principles, from which other results can be deduced formally. What is radically new about this is the fact that the truth of several statement are now based on a formal rather than an experimental principle. Here are Euclid's axioms, loosely translated. These axioms come after some definitions (of a point, line, angle), which we do not reproduce.

(1) It is possible to draw a straight line from any point to any point.
(2) It is possible to extend a finite straight line continuously in a straight line.
(3) It is possible to create a circle with any center and distance (radius).
(4) All right angles are equal to one another.
(5) If a straight line falling on (crossing) two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.
(6) Things which are equal to the same thing are equal to each other.
(7) If equals are added to equals, the wholes (sums) are equal.
(8) If equals are subtracted from equals, the remainders (differences) are equal.
(9) Things that coincide with one another are equal to one another.
(10) The whole is greater than the part.

These axioms are given at the beginning of the treatise and are for use in Geometry.

Books 7, 8 and 9 of the Elements concern Arithmetics but no axioms are given for this subject, which is here to some extent understood as a part of Geometry. Only definitions (of numbers, primes numbers etc.) are given. From Euclid's point of view, Axioms (6)-(10) also apply to numbers so they might be construed as axioms for Arithmetics as well.

After Euclid, there was basically no further intellectual investigation of the axiomatic method before the middle of the nineteenth century. The invention of Calculus in the early seventeenth century led to many interesting mathematical developments over a period of two hundred years but not attempt was made to give an axiomatic presentation of it. From the mid nineteenth century onwards, an axiomatic presentation of Analysis gradually emerges, which subordinates the real numbers to the natural numbers, the so-called "arithmetisation of analysis". From a technical point of view, the following steps are taken :

— Peano's axioms for the natural numbers are introduced, esp. the principle of mathematical induction.
— These are expanded into the Zermelo-Fränkel axiomatic system of set theory, which contains the all-important Axiom of Infinity.

— The ZFC system is sufficient to define the real numbers as the set of Cauchy sequences of rational numbers, modulo an equivalence relation. From there, all the usual analytic notions, eg convergence, differentiation, etc. can be defined.

In particular, Euclidean space is now axiomatically described on the basis of an axiomatic system, whose core describes the properties of the natural numbers. This is a bit odd, because Euclid did not originally provide explicit axioms for Arithmetics, which were subordinate to Geometry. The table are turned now and Geometry is predicated on arithmetic axioms.

This is not the end of the arithmetisation of mathematics, though. In the early twentieth century, formal logic makes its appearance and Arithmetics appear again in the formalisation of metamathematics. The idea is that a first order logical theory can be parameterised by natural numbers, so that a metamathematical first order theory, which describes a first order theory, really makes statements about natural numbers. This was extensively discussed by Mic Detlefsen in his discussion of Gödels 1931 paper. One consequence of this arithmetic description of first order systems is a tightening of the relationship between Arithmetics and the Axiomatic method. Indeed a deduction in a first order system is now described as a sequence of steps, involving modus ponens or generalisation, leading from an axiom of the first order system to a statement and the axioms are now really *axiom schemes*, ie infinite unions of copies of a given axiom with a variable proposition. The axiom schemes are given and not produced from a prime single axiom. A deduction from A to B is now firmly rooted in the principle of induction, with A in place of 0 and B in place of some natural number.

The Arithmetisation of Mathematics is now complete. Not only are all the other subjects axiomatically subordinated to Arithmetics, the very definition of an axiomatic system depends on it. We would like to underline that in Euclid's Elements, the axioms were not, for all practical purposes, used in this way, but rather used as *rules*, ie as general statements that can used in a proof, rather than as the roots of some treelike development from some prime system. To describe this system requires an operation that would certainly not have seemed familiar to Euclid : the enumeration of an infinite number of instances of a certain axiom. The treelike description of axiomatic systems comes from the arithmetic point of view.

This leads to a basic question : is there a description of the foundations of mathematics that is based on the idea of rule rather than on the idea of axiom scheme ? It seems that the Homotopy Type Theory of Voevodsky and others propose something of this kind.

2 Second example. Homotopy.

The second example we want to consider is Homotopy Theory. Homotopy is at the outset a simple topological notion. If X and Y are topological spaces and $f, g : X \to Y$ are continuous maps then one says that f is homotopic to g if there exists a continuous map $[0,1] \times X$, such that $h|_{0 \times X} = f$ and $h|_{1 \times X} = g$. In other words f is homotopic to g if there is a continuous deformation of f into g. In turns out that homotopy is an equivalence relation. It can be used to define many basic concepts of topology. For instance

- — X and Y are homotopy equivalent if there are continuous maps $a : X \to Y$ and $b : Y \to X$ such that $a \circ b$ and $b \circ a$ are homotopic to the identity maps of Y, resp. X. For instance a vector bundle $V \to Y$ is homotopic to Y via the projection map and the 0-section map.
- — X is contractible if the identity map of X is homotopic to the a constant map.
- — X is simply connected if every path (ie continuous map $[0,1] \to X$) is homotopic to any other path via a homotopy, which keeps the end points fixed.

In the work of Quillen on higher K-theory, these topological notions are related to purely categorical notions via the the classifying space functor, which associates a topological space to a category. Under this correspondence, an equivalence of categories $A \sim B$ gives homotopically equivalent classifying spaces. Recall that an equivalence of categories $A \sim B$ is a pair of functors $\phi : A \to B$ and $\psi : B \to A$ such that $\phi \circ \psi$ is isomorphic to the identity functor of B and $\psi \circ \phi$ is isomorphic to the identity functor of A. In the light of this correspondence, an isomorphism of functors can be seen as a kind of homotopy.

This analogy can be pushed to a higher combinatorial level. For instance, there is a notion of homotopy of homotopies, which on the level of categories corresponds to arrows in 2-categories (the category of categories being the prime example of a 2-category). There is a (still not quite stable) notion of ∞-groupoid in topology, which would encode all the paths in a topological space, together with homotopies between them, between the latter and further up, together with increasing complicated compatibility relations. This notion then has a categorical pendant, which is categorical ∞-groupoid.

Now the standard theory of sets ZFC does not seem to be well-adapted to the language of categories. This was already noticed at the time of Quillen's work on K-theory. The basic problem is that objects like 'The category of vector spaces', or 'The category of sets' do not behave like sets, because they are too large for natural set-theoretic operations, like unions or intersections. For a long time, this problem was solved in an ad hoc manner, by replacing the relevant categories by so-called small categories, whose underlying class of objects is a set and which are equivalent to the original category. For example, the full subcategory of the category of $\mathbf{C}$-vector spaces, whose underlying objects are $0, \mathbf{C}, \mathbf{C}^2, \mathbf{C}^3, \ldots$ is such a subcategory.

It seems that the Homotopy Type Theory of Voevodsky and others provides a new language for the foundations of mathematics, which extends ZFC and draws on ideas from homotopy to solve the above problem. Here are some of its salient features :

- the basic object of the theory is the type ; every statement states that some type is of some type.
- sets can be construed as 'discrete' types whereas categories, but also more generally n-categories, will be described by more general types ;
- characteristically, every type should have an associated '∞-groupoid' ;
- the theory has no foundational axioms but only rules of type formation.

So HoTT moves away from ZFC at the same time as from first order logic. It moves aways from ZFC in the sense that it is not constructive as far as its objects are concerned : the types are not "inductively" constructed from the empty set, they are simply a given part of the language. It also moves away from first order logic in the sense that it is not propose axioms but rules. HoTT thus breaks away completely from the arithmetical take on mathematics that has been familiar since the early twentieth century.

3 Other candidates

• Algebraic Duality Theories (Poincaré duality, Serre duality)

• The Fourier transform (Pontrjagin duality, Fourier-Mukai, Orlov's theorem and its generalisations).

• Riemann-Roch (Hirzebruch-Riemann-Roch, Grothendieck-Riemann-Roch, Index theorems, functorial Riemann-Roch...).

4 The meaning of mathematical theories. Symbolic forms.

Arithmetics as a symbolic form - see my talk 'L'arithmétique comme forme symbolique'.

Homotopy as a symbolic form :

Expressive stage. The equivalence of physical shapes that can be deformed into each other.

The ’generic’ drawings.

Representational stage. The mathematical definition of a topological homotopy.

Semantic stage. HoTT